Bao - I

Goal: construct

- M^{∇}_{to} algebraic part of EG stack

Motivation: Weight part of Serre's conjecture & Breuil Mézard conj.

Serre's conjectures: $\bar{\rho}: G_{\mathbb{Q}} \to GL_2(\bar{\mathbb{F}}_p)$ irr., unramified a.e., odd.

then $\bar{\rho} = \bar{\rho}_{f,p}$ → Associated Gal. rep^n of $f \in S_k(\Gamma_1(N))$ eigenform.

Stronger form: Describe N, k you can take.

Recipe for possible $N = N(\bar{\rho})$ = prime to p Artin conductor

Weight k: can take k = explicit combinatorial n.b. from $\bar{\rho}|I_{\mathbb{Q}_p}$ ← wild ramification

$k < p+1$ if allow twists by ε

e.g. $\bar{\rho}|I_{\mathbb{Q}_p} \simeq \begin{pmatrix} \omega^{k-1} & * \\ 0 & 1 \end{pmatrix} \rightsquigarrow$ weight k $\left(\begin{matrix} \text{condition of } k? \\ * \text{ non split} \end{matrix}\right)$

p-adic HT origin: for small k ($<p+1$), constraints on $\bar{\rho}_{f,p}|I_{\mathbb{Q}_p}$ for $f \in S_k(\Gamma_1(N))$

e.g. $\rho_{f,p}$ crystalline with HT w. $(k-1, 0)$

$\left(\bar{\rho}_{f,p}^{ss}|I_{\mathbb{Q}_p} = \omega^{k-1} \oplus 1 \text{ if } \bar{\rho}_{f,p}|G_{\mathbb{Q}_p} \text{ is reducible}\right)$

(2)

if put so, where is rep. theory? || where is used. ?

e.g.

$\bar\rho$ reducible : $\bar\rho'|_{I_{\mathbb{Q}_p}} = \begin{pmatrix} \omega^{k-1} & \\ & 1 \end{pmatrix}$

(This is special to reducible.)

$\bar\rho'|_{I_{\mathbb{Q}_p}} \otimes \omega^{1-k} = \begin{pmatrix} 1 & \\ & \omega^{1-k} \end{pmatrix}$ can take "$k = p-k+1$" after twisting

Reformulation : (In fact, can start with $\bar\rho = \bar\rho_{f,p}$, $f \in S_k(\Gamma_1(N))$

Then Serre recipe gives $f \equiv f' \in S_{k'}(\Gamma_1(N))$.)

The story so far : data from f you just care of $\phi : \mathbb{T} \to \overline{\mathbb{F}}_p$

mod p system of Hecke eval

Hecke ms $S_k(\Gamma_1(N)) \hookrightarrow H^1(Y_1(N), \mathrm{Sym}^{k-2}\mathbb{C}^2)$

$\mathbb{T}$- equivariant

all info of ϕ shows up here

$\bar\rho_f \mapsto \bar\rho_\phi$ Question: for which k does ϕ contribute to $H^1(Y_1(N), \mathrm{Sym}^{k-2}\mathbb{C}^2)$?

$\mathbb{T} \circlearrowright$; $\cup$ $\mathrm{Sym}^{k-2}\overline{\mathbb{Z}}^2$

& $\bar\rho = \bar\rho_{f,p}$ $\exists f \in S_k(\Gamma_1(N)) \Leftrightarrow H^1(Y_1(N), \mathrm{Sym}^{k-2}\overline{\mathbb{Z}}_p^2)_{\mathfrak{m}} \neq 0$ $\mathfrak{m} = \mathrm{Ker}(\phi : \mathbb{T} \to \overline{\mathbb{F}}_p)$

$\Updownarrow$

$\underline{\qquad\qquad} \overline{\mathbb{F}}_p^2)_{\mathfrak{m}} \neq 0$ if $\bar\rho$ is irred.

More generally, $Y_1(N) = GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / K_\infty \cdot GL_2(\mathbb{Z}_p) K_1^p$

For any $V \circlearrowleft \mathcal{O}[GL_2(\mathbb{Z}_p)]$ Define $\rightsquigarrow \left((GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A})) \times V \right) / K_\infty \cdot GL_2(\mathbb{Z}_p) \cdot K_1^p$ $\leftarrow$ prime to p level

($\mathcal{O}$ = ring of integers of $E/\mathbb{Q}_p < \infty$)

~~Pb~~ that SWC solves:

For which $\mathcal{O}[GL_2(\mathbb{Z}_p)]$-module V, $H^1(Y_1(N), V)_{\mathfrak{m}} \neq 0$?

$\rightsquigarrow K_0(\overline{\mathbb{F}}_p[GL_2(\mathbb{Z}_p)]\text{-mod}) = \bigoplus_{\substack{2 \leq k \leq p+1 \\ 0 \leq \ell \leq p-1}} \mathbb{Z}[\mathrm{Sym}^{k-2} \overline{\mathbb{F}}_p^2 \otimes \det^\ell]$

Serre's recipe: $H^1(Y_1(N), \mathrm{Sym}^{k-2} \overline{\mathbb{F}}_p^2 \otimes \det^\ell)_{\mathfrak{m}} \neq 0$ iff k "compatible" with $\bar\rho|_{I_{\mathbb{Q}_p}}$

$$\|$$

$$H^1(Y_1(N), \underbrace{\mathrm{Sym}^{k-2} \mathbb{Z}_p^2 \otimes \det}_{\substack{\downarrow \\ \text{irr loc.} \\ \text{system in char } 0}})_{\mathfrak{m}} / p$$

④

<u>In general</u>

$G_{/F}$ reductive
unram. at p

$K = G(\mathcal{O}_{F_p})\, K^p$

$$Y_K = G(F)\backslash G(\mathbb{A}_F) / K_\infty \cdot G(\mathcal{O}_F) \cdot K^p$$

$\mathcal{O}[G(\mathcal{O}_F)]$-mod $\longrightarrow$ Complexes with $\mathbb{T}$-action

$$V \longmapsto R\Gamma(Y_K, V)_{\mathfrak{m}}$$

Guiding principles:

$K_0(\overline{\mathbb{F}}_p[G(\mathcal{O}_{F_p})]) = \bigoplus \mathbb{Z}\,[\sigma]$

irr rep of $G(\mathbb{F}_p)$ over $\overline{\mathbb{F}}_p$

system of mod p Hecke eval attached to $\bar{\rho}: G_F \to GL_n(\overline{\mathbb{F}}_p)$

① List $W(\phi) = \{\sigma \text{ s.t. } R\Gamma(Y_K, \sigma)_{\mathfrak{m}} \neq 0\}$

should depend only on the collection $\{\rho\, \phi|_{I_{F_v}}\}_{v|p}$!

② Constraints from p-adic HT should be necessary & sufficient.

We need a context where $R\Gamma(Y_K, V)_{\mathfrak{m}}$ concentrates in 1 degree (i.e. is nonzero in only one degree)

E.G.:

$G = D^\times$ ⇝ BDJ answer via the following principle: Key: any SW of GL_2 lifts to char 0!

= definite Serre weights (i.e.) → Generalized by Herzig / GHS

Key difficulty (for GL_n, $n \geq 3$): [mod p σ] (i.e. Serre Weights) don't lift in char 0!

can't easily lift p-adic HT condition!

$\exists$ only one direction: Set $M(\sigma) := H^0(Y_K, \sigma)_{\mathfrak{m}}$ (G definite)

↳ if $M(\sigma) \neq 0 \Rightarrow M(V) \neq 0$ $\forall V$ irred, $\sigma \in JH(\bar{V}^{\circ})$.

V: $GL_n(\mathbb{F}_p)$-rep$/\mathcal{O}$

GHS conj: requires the converse is true

$$\left\{ \text{Irrep. } GL_n(\mathbb{F}_p) \right\} \xleftrightarrow{\sim} X_1^*(T) \big/ (p-1)X_0^*(T)$$

$L(\lambda)|_{GL_n(\mathbb{F}_p)}$

with $X_1^*(T) = \{ 0 \leq \langle \lambda, \alpha^\vee \rangle \leq p-1 \ \forall \alpha \text{ simple root} \}$

$$\left\{ \text{Irrep } GL_n/\overline{\mathbb{F}}_p \right\} \longleftrightarrow X^*(T)_{dom}$$

dominant weights

$$L(\lambda) \longleftrightarrow \lambda$$

Here $L(\lambda) = \mathrm{soc}\, \underbrace{W(\lambda)}_{\mathrm{Ind}_B^G w_0\lambda}$.

(Algebraic)-Weyl module

GHS conj: $\underset{L(\lambda)|_{G(\mathbb{F}_p)}}{\overset{\shortparallel}{\sigma}} \in W(\bar\rho)$ $\boxed{\text{iff}}$ $M(W(\lambda)|_{GL_n(\mathbb{Z}_p)}) \neq 0$

iff $\bar\rho_p$ has crystalline lifts of HT wt. $\lambda + \eta$

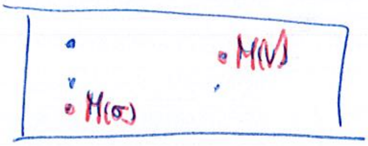
Note that ∃ many relations $K_0(G(\mathbb{Z}_p)) \twoheadrightarrow K_0(G(\mathbb{F}_p))$

what
Can do: $[\sigma] = \sum n_V [\overline{V}]$ in $K_0(GL_n(\mathbb{F}_p))/\overline{\mathbb{F}}_p$, $n_V \in \mathbb{Z}$ $\quad$ V rep. of $GL_n(\mathbb{Z}_p)$ in char 0, irred.

Can look at $\rightsquigarrow$ $\mathrm{length}(M(\sigma))$ "=" $\sum n_V \cdot \mathrm{length}(M(\overline{V}))$ $\quad$ (Q. why "=" what "cancellations")

$\circlearrowleft$

$R_{\overline{\rho}\phi}$ global Gal. def.



better invariant: supp. of cycles

<u>Solution</u>/<u>Improvement</u>: $M(\sigma) \rightsquigarrow M_\infty(\sigma)$

$\circlearrowleft$ $\quad$ TW $\quad$ $\hat{\otimes}_{v|p} R^{\square}_{\overline{\rho}\phi|G_{F_v}}$ $[[x_1,\dots,x_q]]$

$R_{\overline{\rho}\phi}$

<u>Key</u>: 1. V free $\mathcal{O}$-mod. (or $\mathbb{F}$-mod) $\quad$ (Q. Sheaf on $\mathcal{X}_{EG}$)

maximal Cohen-Macaulay
$M_\infty(V)$ is MCM over its support

2. $M(\sigma) \neq 0 \Leftrightarrow M_\infty(\sigma) \neq 0$

3. If V is $W(\lambda) \otimes \sigma(\tau)$ then $\mathrm{Spec}(R^{\lambda+\eta,\tau}_{\overline{\rho}|G_{F_v}}) \hookrightarrow \mathrm{Spec}(R^{\square}_{\overline{\rho}|G_{F_v}})$

inertial local Langlands
HT $\lambda+\eta$
inertial type τ

$\mathrm{Supp}(M_\infty(V))$

⑦

$M_\infty(\nabla)$: " has equidim'd support " & MCH

$Z(M_\infty(\sigma)) \neq 0$ iff $\sum n_\nabla Z(M_\infty(\nabla)) \neq 0$

here → "Better chances for cancellations"

$C_\sigma^{out} = \sum n_\nabla C_{\nabla}^{out}$ left to understand → just to understand $C_\sigma^{out} = M_\infty(\sigma \otimes \sigma)$

Conj: $C_{w \otimes \sigma \otimes \sigma} = Z(R^{\lambda+\eta, \tau}/p)$

This implies C_σ^{out} is local

follows from $R^{\lambda+\eta,\tau}$ being a domain

Rmk: Many choices of $\sum n_\nabla [\nabla] = \sigma$

$\Downarrow$

[Here relations btw $R^{\lambda+\eta,\tau}/p$] ← BM conj

Bao-II

Reasons to study $R_{\bar{\rho}}^{\lambda,\tau}$

Strategy: 2 steps:

$$\begin{tikzcd}
\{(\rho, \mathfrak{M})\} \arrow[r, hook, "\text{iso if } \lambda \text{ small}"] \arrow[d] & \{\mathfrak{M}\} \arrow[d] \\
\{\rho \text{ of type } (\lambda,\tau)\} \arrow[r] & \Phi\text{-Mod}
\end{tikzcd}$$

BR module of type λ τ

The difference is in the $\exists$ of monodromy operator

iso in char 0

i.e. λ = minuscule

iso in "generic" situation

Goal: Analyze $\{\mathfrak{M}\}$

$$\begin{tikzcd}
& \{\mathfrak{M}\} & \\
\{\bar{\rho}, \mathfrak{M}\} \arrow[ur] & & M(\lambda) \arrow[ul] \\
& M(\lambda)^{\nabla} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$M(\lambda)$ = mixed char. Schubert variety

$K = \mathbb{Q}_p$ (in general $[K:\mathbb{Q}_p] < \infty$ & small ramification) Coefficients: $E \supseteq \mathcal{O}_E$

$K_\infty = \bigcup_n K\left((-p)^{\frac{1}{p^n}}\right)$; τ: tame inertial type

(In fact, will assume τ mean 1)

$L = K(\zeta_p)$

$L_\infty = \bigcup L\left((-p)^{\frac{1}{ep^n}}\right)$ $e = p-1$

$GL_n, B, T, X_*(T), X^*(T), \Phi$ roots

α_i: simple root

②

$u^\lambda := \mathrm{Diag}(u^{\lambda_1}, \dots, u^{\lambda_n})$

$W = W(GL_n) = S_n$ | acts simply transitively on Alcoves

$\widetilde{W} = X_*(T) \rtimes W$ | $\widetilde{W} \circlearrowright X_*(T) \otimes \mathbb{R}$

$\cup|$ $\quad\cup|$

$\widetilde{W}_a = R \rtimes W$ | Alcove: fund. domain for the action

R ↑ root lattice

τ: tame inertial type

$$\begin{tikzcd} \tau : I_{\mathbb{Q}_p} \arrow[r] \arrow[dr, "\bar\tau"'] & GL_n(\mathcal{O}) \arrow[d] \\ & GL_n(\mathcal{O}/\varpi) \end{tikzcd}$$

classified by $(s,\mu) \in \underbrace{W \times X_*(T)}_{\widetilde{W}}$

$(s,\mu) \mapsto$ étale φ-module with $\mathrm{Mat}(\varphi) = s \cdot v^\mu = \widetilde{\omega}$

$\mathcal{O}_L = \mathbb{Z}_p[[u]]$ $\qquad v = u^e$

$\mathcal{O}_K = \mathbb{Z}_p[[v]]$

Redundancies: $s \cdot v^\mu \sim \widetilde{\omega}^{-1} s v^\mu \varphi(\widetilde{\omega}^{-1})$ $\quad$ φ-twisted (change of basis) $\quad \widetilde{\omega} = w t_\nu$

e.g. $s=1$, $v^\mu \sim w v^\nu v^\mu v^{-p\nu} w^{-1}$

$\Rightarrow$ can find representatives of the form $(1,\mu)$, $0 < \langle \mu, \alpha \rangle < p-1$ $\forall \alpha \in \Phi$

Say μ weakly generic if $? \leq |\mu_i - \mu_j| \leq p - ?$ for $? = ?(n)$

③ $\lambda \in (X_*(T))_{\text{dom}}$
Fix effective

$\Delta = \mathrm{Gal}(L/K) \simeq \mu_{p-1}$
$\mathfrak{S}_L : \sigma(u) \overset{\text{def}}{=} \omega(\sigma)\cdot u$

Def: BK module over A of type (λ, τ) is a BK module for $\mathfrak{S}_L(A \otimes_{\mathbb{Z}_p} \mathbb{Z}_p)[[u]]$ bounded by λ projective, of rk d & action of Δ,

- Δ semilinear on $\mathfrak{M}\mathfrak{S}_{L,A}$
- φ, Δ commute
- Bounded by λ: $(E(u))^{\sum_{j=1}^{i} \lambda_j}$ for all $i = 1, \dots, n$ | all i by i minors of $\mathrm{Mat}_\beta(\varphi)$ eigenbasis of $\mathfrak{M}$
- $\mathfrak{M}/u\mathfrak{M} \underset{\text{locally}}{\simeq} \tau \otimes_{\mathbb{Z}_p} A$ $\circlearrowleft \Delta$

Goal: moduli of such $\mathfrak{M}$:

locally $\mathfrak{M}$ has a basis β, with β eigenvectors for Δ-action

i.e. $\delta \cdot \beta = \beta \cdot (\omega(\delta))^{\mu}$ → if $\tau = \tau(1, \mu)$

$\rightsquigarrow C_\beta = \mathrm{Mat}_\beta(\varphi_{\mathfrak{M}}) \in GL_n\left(A[[u]]\left(\frac{1}{E(u)}\right)\right)$

$\in \mathrm{Mat}_n(A[[u]])$

& $\mathrm{Ad}(u^{\mu})\, C_\beta =: A_\beta \in \mathrm{Mat}(A[[v]])$

$\overset{\text{"}}{u^{\mu} C_\beta u^{\mu}}$

$L^+\mathcal{G}(A) = \{X \in GL_n(A[[v+p]]),\ X \bmod v \text{ is upper triangular}\}$

$L\mathcal{G}(A) = \{X \text{ ——— } A((v+p)),\ \text{———————}\}$

$L^-\mathcal{G}(A) = \{X \in L\mathcal{G}(A[\tfrac{1}{v+p}]),\ X \bmod v \text{ is unipotent upper } \nabla \bmod \tfrac{1}{v+p}\}$

Met

Note: $A_\beta = (\mathrm{Ad}(v^{-\mu}))\varphi_\beta \in L\mathcal{G}(A)$

$\rightsquigarrow$ Get:

$$(\mathcal{M}, \text{type } (\lambda, \tau)) \longrightarrow {}^{L\mathcal{G}^{\leq\lambda}}\!/\!_{L^+\mathcal{G}}\ (\varphi, \tau)$$

assume μ regular ie. $0 \ll \langle \mu, \alpha \rangle \ll p-1\ \forall \alpha \in \Phi$

$\underbrace{(\varphi, \tau)}$

φ-twisted conj:

ie. $A_\beta \sim X A_\beta (\mathrm{Ad}(v^\mu)(\varphi(X^{-1})))$

<u>Prop</u>: (Assume τ "weakly generic")

$$\left[{}^{L\mathcal{G}^{\leq\lambda}}\!/\!_{L^+\mathcal{G}}\ (\varphi,\tau) \right]_{\mathbb{F}} = \left[{}^{L\mathcal{G}^{\leq\lambda}_{\mathbb{F}}}\!/\!_{L^+\mathcal{G}_{\mathbb{F}}}\ (\varphi,\tau) \right] =$$

$$= \left[{}^{LGL_n{}^{\leq\lambda}_{\mathbb{F}}}\!/\!_{I}\ (\varphi,\tau) \right] = \underbrace{\left[{}^{LGL_n{}^{\leq\lambda}_{\mathbb{F}}}\!/\!_{I_1} \Big/ T \right]}$$

T-torsor over $\mathrm{Fl} = {}^{LGL_n{}^{\leq\lambda}_{\mathbb{F}}}\!/\!_{I}$

Pf: $B = X A \,\mathrm{Ad}(v^{\mu})\, \varphi(X^{-1}) \qquad X \in I_1$

$\rightsquigarrow X A\, \mathrm{Ad}(v^{\mu})\, \varphi(X^{-1}) A^{-1} \in I_1$

Warning: <u>false if $p \neq 0$ in A</u>:

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & v+p \end{pmatrix}\begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (v+p)^{-1} \end{pmatrix} \notin I_1$$

But the iso lifts locally:

$$\begin{array}{c} \widetilde{\mathcal{Fl}} \\ \Big\downarrow{\scriptstyle T} \\ \mathcal{Fl} = \bigcup_{\widetilde{\omega}} \underbrace{I \widetilde{\omega} I / I}_{S(\widetilde{\omega})} \xrightarrow{\text{descends to}} \left[\widetilde{\mathcal{Fl}} / T\right] \end{array}$$

Fact: $L^{-}\mathcal{G} \times L^{+}\mathcal{G} \to L\mathcal{G}$ open immersion

$\rightsquigarrow$ open cover of $\mathcal{Fl} = \bigcup_{\widetilde{\omega}} (L^{-}\mathcal{G})_{\mathbb{F}} \cdot \widetilde{\omega}$ open cover

Def: A BK module $\mathfrak{M}$ of type (λ, τ) admits a $\widetilde{\omega}$-gauge if $\mathfrak{M}$ occurs in $(L^{-}\mathcal{G})_{\mathbb{F}}\,\widetilde{\omega}$

Let $\tilde{\omega} = w t_\nu$

$$\mathcal{U}(\tilde{\omega})(R) = \left\{ A \in L\mathcal{G}(R) \cap GL_n(R((v+p))) \ , \ A(v+p)^{-\nu} w^{-1} \in GL_n\left(R\left[\tfrac{1}{v+p}\right]\right) \text{ unip. lower triangular mod } v+p \right\}$$

$$\Updownarrow$$

$$A(v+p)^{-\nu} \in GL_n\left(R\left[\tfrac{1}{v+p}\right]\right) \text{ upper } \nabla \text{ mod } \frac{v}{v+p}.$$

e.g. $\tilde{\omega} = \begin{pmatrix} & 1 \\ v^2 & \end{pmatrix}$

$$\mathcal{U}(\tilde{\omega}) = A = \begin{pmatrix} c(v+p) + c' + \frac{c''}{v+p} + \cdots & 1 + \frac{c'''}{v+p} + \cdots \\ v\left((v+p) + c' + \frac{c''}{v+p}\right) & \frac{c'}{v+p} + \frac{c''}{(v+p)^2} + \cdots \end{pmatrix}$$

<u>Lemma</u>: $\exists$ canonical iso

$$\left\{ \begin{array}{l} \mathfrak{M}, \text{ type } (\tau, \lambda) \\ \overline{\mathfrak{M}} \text{ admits } \tilde{\omega}\text{-gauge} \end{array} \right\}^{\wedge_p} \xrightarrow{\sim} \left[T\,\mathcal{U}(\tilde{\omega}) / T \right]^{\wedge_p}$$

p-adic completion

i.e., any such $\mathfrak{M}$ over R with $pR = 0$ admits a basis β s.t. $A_\beta \in \mathcal{U}(\tilde{\omega})$

unique up to T

⑦

Pf: Reduce to lifting β; boils down to solving equations

$$U + X - \mathrm{Ad}(A\sigma^{\mu})(\cancel{X}) = Z$$

$U \in \mathrm{Lie}\, L^{-}G$, $X \in L^{+}G$, $Z \in (\mathrm{Lie}\, L^{-}G)_{F}$

Bao - III

$K = \mathbb{Q}_p$, $\lambda \in X_*(T)_{\text{dom}}$, τ tame inertial type

$$\updownarrow$$

presentation $\to$ $(s, \mu) \in W \times X^*(T)$

$$\updownarrow$$

$$\tilde{\omega}_\tau = s^{-1} t_\mu \in \widetilde{W} \hookrightarrow Fl = I \backslash (LGL_n)_{\mathbb{F}}$$

μ "minimal"

e.g. $s=1$, $0 \leq \mu_i - \mu_j \leq p-1$

$$\begin{matrix} \varphi & \Delta \\ \circlearrowleft & \circlearrowleft \end{matrix}$$

$$\left\{ \begin{matrix} \text{BK module } \mathfrak{M} \text{ for } \mathcal{O}_K \\ \text{of type } (\lambda, \tau) \end{matrix} \right\} =: \mathcal{C}_{n, \leq\lambda, \tau}$$

By choosing bases locally $\mathcal{C}_{n,\leq\lambda,\tau} = \left[L^+\mathcal{G} \diagdown L\mathcal{G}_{(\varphi,\tau)} \right]$ $\quad$ $L\mathcal{G}$, $L^+\mathcal{G}$ twisted loop group

the p-adic completion is a formal p-adic alg. stack

$$\mathcal{C}_{n,\leq\lambda\tau\,\mathbb{F}} = \left[L^+\mathcal{G} \diagdown L\mathcal{G}_{\varphi,\tau} \right]_{\mathbb{F}} = \left[\widetilde{Fl} / T\text{-conj} \right]$$

can glue in $\neq$ ways in char p

$$\| \qquad\qquad\qquad \|$$

$$\bigcup \mathcal{C}_{n,\leq\lambda,\tau}(\tilde{\omega})_{\mathbb{F}} \qquad \bigcup_{\tilde{\omega}} \left[{}^{T}\mathcal{U}(\tilde{\omega})_{\mathbb{F}} / T\text{-conj} \right]$$

Lemma: $\exists$ canonical lift $\mathcal{C}_{n,\leq\lambda\tau}(\tilde{\omega}) \xrightarrow{\sim} \left[{}^{T}\mathcal{U}(\tilde{\omega}) / T \right]^{\wedge p}$

② & $Gr_E^{\leq\lambda} = S(\lambda)$

Here

glue $\mathcal{U}(\tilde{\omega})_{\mathbb{F}}$ in $\mathcal{F}\ell$

$\downarrow$

I can glue $\mathcal{U}(\tilde{\omega})$ to get $L^+\mathcal{G}\backslash L\mathcal{G}^{\leq\lambda} = Gr^{\leq\lambda}$

with $Gr_E^{\leq\lambda} = Gr_{GL_n}^{\leq\lambda}$, $Gr_{\mathbb{F}}^{\leq\lambda} = \mathcal{F}\ell^{\leq\lambda}$.

[in particular, have this map]

$$\begin{tikzcd}
\mathcal{C}_{n,\leq\lambda,\tau}(\tilde{\omega}) \arrow[r, "\sim"] \arrow[d, hook] & {[T\mathcal{U}(\tilde{\omega})^{\leq\lambda}/T]^{\wedge_p}} \arrow[d, leftrightarrow] \arrow[dr] & \\
\mathcal{R}_d & A \cdot s' v^{\mu} & {[Gr^{\leq\lambda}/T]^{\wedge_p}}
\end{tikzcd}$$

monomorphism if τ is suff. generic

Emerton-Gee φ-modules

A

Fact (Zhu): $M(\lambda) \overset{def}{=}$ flat closure of $S(\lambda)$ in Gr. Then $M(\lambda)_{\mathbb{F}} = \bigcup_{\tilde{\omega} \in Adm(\lambda)} I\tilde{\omega}I$

$\bar{\rho} : G_K \to GL_n(\mathbb{F}_p) \rightsquigarrow R_{\bar{\rho}}^{\leq\lambda,\tau} \neq 0 \Rightarrow \exists\, \overline{\mathfrak{M}}_{\bar{\rho}} \in \mathcal{C}_{n,\leq\lambda,\tau}$

pt. giving rise to $\bar{\rho}|_{G_{K_\infty}}$

③ Get

Framed deformation ring with HT $W \leq \lambda + \eta$ type τ

$$\left[\mathrm{Spf}\, R_{\bar\rho}^{\leq\lambda,\tau} \Big/ GL_n\right] \xhookrightarrow{\iota} \mathcal{C}_{n,\leq\lambda,\tau}(\tilde\omega) \xrightarrow{\sim} \left[TU(\tilde\omega)^{\leq\lambda} / T \right]^{\wedge_p}.$$

monomorphism $\dashrightarrow R_d$

Corollary: If $R_{\bar\rho}^{\leq\lambda,\tau} \neq 0$ then $\tilde\omega_{\bar\rho^{ss}}|_{I_{\mathbb{Q}_p}} \in \mathrm{Adm}(\lambda)\, \tilde\omega_\tau$ (Hypotheses: λ regular, τ generic relative to λ)

Rmk: 1. If $\bar\rho = \bar\rho^{ss}$ then have "$\Leftrightarrow$"
(only obstruction to lift $\bar\rho$ to a rep. of type (λ,τ) is lifting its BK module)

2. True also for $\bar\rho \neq \bar\rho^{ss}$

3. (1) is strong enough to show $W(\bar\rho_\phi) \subseteq W^{GHS}(\bar\rho_\phi)$

Need to understand what lift of BK modules appear in Im(ι)

$\mathfrak{M} \mapsto \mathfrak{M} \otimes \mathcal{O}^{rig}[\frac{1}{\lambda}]$ has canonical N derivation: $-\lambda u \frac{d}{du}$ on $\mathcal{O}^{rig}$,

$N\varphi = E(u) \varphi N, \quad N \equiv 0 \bmod u$

If β basis of $\mathfrak{M}$

$$\mathrm{Mat}_\beta(N) = \left(\lambda u \frac{d}{du} C_\beta\right) C_\beta^{-1} + E(u)\, C\, \varphi\left(\left(\lambda u \frac{d}{du} C_\beta\right) C_\beta^{-1}\right) C^{-1} + \dots$$

Condition: N preserves $\mathfrak{M} \otimes \mathcal{O}^{rig}$ i.e.

$\rightsquigarrow$

$$= \lambda e\left(v \frac{d}{dv} A_\beta\right) A_\beta^{-1} \boxed{+ [\mu, A_\beta] A_\beta^{-1}} + (\dots) \in \mathcal{O}^{rig}$$

$+\mu \quad +\mu A$

④ A priori the expressions have poles at $\vartheta = -p$!

Condition: $\left(\frac{\vartheta d}{d\vartheta} A\right) A^{-1} + (\ldots)$ has pole of order ≤ 1 at $\vartheta = -p$

$+[\mu, A]$ $\quad$ p in denominators !

Get $\underbrace{\left(\frac{d}{d\vartheta}\right)^s \left(\left(\frac{\vartheta d}{d\vartheta} A\right) A^{adj} + \ldots\right)_{\vartheta = -p}}_{\text{this expression makes sense integrally !}} = 0 \quad \forall\, s = 0, \ldots, ?-1 \quad (*)$

(where $\det A = (\vartheta + p)^{?}$)

Def: $(T\mathcal{U}(\tilde{\omega}))^{\nabla_{true}} = \{A \text{ s.t. } (*) \text{ is true}\}^{p\text{-flat}}$

$(T\mathcal{U}(\tilde{\omega}))^{\nabla_{naive}} = \{A \text{ s.t. } \left(\frac{\vartheta d}{d\vartheta} A\right) A^{-1} + \frac{1}{e}[\lambda, A] A^{-1} \text{ has poles of order} \leq 1 \text{ at } \vartheta = -p\}$

$(T\mathcal{U}(\tilde{\omega}))^{\nabla} = \left((T\mathcal{U}(\tilde{\omega}))^{\nabla_{naive}}\right)^{p\text{-flat}}$

$$\begin{tikzcd}
 & \left[T\mathcal{U}(\tilde{\omega})^{\nabla_{true}}/T\right] \arrow[d] \\
\left[\mathrm{Spf}\, R_{\bar{\rho}}^{\leq\lambda,\tau}/GL_n\right] \arrow[ur, "\sim"] \arrow[r, hook, "?"] & \left[T\mathcal{U}(\tilde{\omega})^{\leq\lambda}/T\right]^{\wedge_p}
\end{tikzcd}$$

Goal: Understand $(T\mathcal{U}(\tilde{\omega}))^{\nabla_{true}} \equiv (T\mathcal{U}(\tilde{\omega}))^{\nabla_{naive}}$ $\boxed{\bmod\ p^{\max|\langle\mu, \alpha_i^\vee\rangle| - C(n,\lambda)}}$

Constant $\in \mathbb{N}$ depending on n, λ

<u>Thm</u> : 1 Assume τ is generic w.r.t. λ. Then $\exists$ <u>noncanonical</u>

lift

$$\left[T\mathcal{U}(\tilde{\omega})^{\leq\lambda\,\nabla_{\text{tue}}}/T\right] \hookrightarrow \left[T\mathcal{U}(\tilde{\omega})^{\leq\lambda\,\nabla}/T\right]^{\lambda_p}$$

of the $\left[T\mathcal{U}(\tilde{\omega})^{\leq\lambda\,\nabla_{\text{tue}}}/T\right]_{\mathbb{F}} \hookrightarrow \left[T\mathcal{U}(\tilde{\omega})^{\leq\lambda\,\nabla_{\text{naive}}}/T\right]_{\mathbb{F}} \hookrightarrow \left[T\mathcal{U}(\tilde{\omega})^{\leq\lambda}/T\right]_{\mathbb{F}}$

2 If $\boxed{\mu \bmod p \text{ avoid } \cancel{\text{some}} \text{ Zariski closed set of } \mathbb{F}_p^n \text{ (depending on } \lambda, \mu)}$ $\leftarrow$ "strong generacity of $(\lambda,\tilde{\omega})$" then the above map is iso on max^d dim^n part.

<u>Rmk</u> : 1 If $\bar{\rho} \leftrightarrow x$ and complete at x, have

$$\left[\mathrm{Spf}\, R_{\bar{\rho}}^{\leq\lambda,\tau}/GL_n\right] \xrightarrow{\sim} \left[T\mathcal{U}(\tilde{\omega})^{\leq\lambda\,\nabla}/T\right]^{\lambda_x} \text{ on max}^d \text{ dim}^d \text{ component}$$

2. $\lambda=\eta$ $\left[\mathrm{Spf}\, R_{\bar{\rho}}^{\eta,\tau}/GL_n\right] \simeq$ max^d dim^d part of $\left[T\mathcal{U}(\tilde{\omega})^{\leq\eta\,\nabla}/T\right]^{\lambda_x}$

3. Only assume τ is generic w.r.t. λ then $\left[T\mathcal{U}(\tilde{\omega})^{\leq\lambda\,\nabla_{\text{tue}}}/T\right]_{\mathbb{F},\text{red}} = \left[T\mathcal{U}(\tilde{\omega})^{\leq\lambda\,\nabla_{\text{naive}}}/T\right]_{\mathbb{F},\text{red}}$

4 These glue into $(\mathcal{X}_{n,\leq\lambda,\tau})_{\text{red}} = \left[\tilde{M}(\lambda)_{\mathbb{F}}^{\nabla_{\text{naive}}}/T\right]_{\text{red}}$

Recall $T\mathcal{U}(\tilde{\omega})^{\leq\lambda\,\nabla_{\text{naive}}} = \left\{A \in T\mathcal{U}(\tilde{\omega})^{\leq\lambda} \text{ s.t. } \left(v\frac{d}{dv}A\right)A^{-1} + [\cancel{\mu}, A]A^{-1} \text{ has poles of ord} \leq 1 \text{ at } v=-p\right\}$

⑥ descends to $\mathcal{U}(\tilde{\omega})^{\leq\lambda,\nabla\text{naive}} \subseteq M(\lambda)$

& $M(\lambda)^{\nabla nv} = \bigcup \mathcal{U}(\tilde{\omega})^{\leq\lambda,\nabla nv} = \{A \in M(\lambda), \text{ s.t.} \dots\}$

Note: without the term $\left(\frac{\sigma d}{d\sigma} A\right)A^{-1}$ have an affine Springer fiber.

Sketch of pf of theorem.

Item 1. Lift via Elkik's lemma: A p-adically complete flat $\mathbb{Z}_p$-algebra

$$B = A[T_1, \dots, T_n] \big/ (f_1, \dots, f_m)$$

$$H_B = \text{Jacobian of } B/A \qquad h \geq 1$$

Then if $(a_1, \dots, a_n) \in A^n$ with $f_1(\underline{a}) = 0$, $\vdots$, $f_m(\underline{a}) = 0$ mod p^h

$\exists \varepsilon \leq h$

& $p^{h/2} \in H_B(\underline{a})$ then can lift $\underline{a}$ mod $p^{h-\varepsilon}$ to a "true" solution $\tilde{\underline{a}} \in A^n$

The main theorem follows from:

$$M(\lambda)^{\nabla nv}\left[\tfrac{1}{p}\right] = \bigsqcup_{\substack{\lambda' \leq \lambda \\ \text{dominant}}} P_{\lambda'} \backslash GL_n$$

i.e. gen. fiber is smooth

ie. ~~comps~~ Irred. comp. of completion of $(M(\lambda)^\triangledown)^{\wedge}_{\varpi}$ ~~in~~ bijection with irred. components of $\underbrace{(M(\lambda)^\triangledown)_{\varpi}}$

⑦

$M(\lambda)^\triangledown$ is unibranch at each $\varpi \in M(\lambda)^\triangledown_{\mathbb{F}}$ ie.

$\underline{\qquad\qquad\qquad\qquad}$ $\varpi \in M(\lambda)_{\mathbb{F}}$ ie.

$\underline{\qquad\qquad\qquad\qquad}$ $\varpi \in \mathrm{Adm}(\lambda)$.

Boo 4

$\lambda \in X_*(T)_{\mathrm{dom}}$ $\quad \tau = R_s(\mu)$ $\quad \tilde{\omega}_\tau = s^{-1} t_\mu \in \widetilde{W}$

Thm: This is iso mod lower dim'l components

Have such map, mod high power of p $\Rightarrow$ use El_{KR} to lift in class $\mathcal{O}$ in an immersion

$$\begin{tikzcd}
\widetilde{X}_{n,\leq\lambda,\tau} \arrow[r] \arrow[d, "p\text{-adic formal stack}"] \arrow[rr, bend left] & {[T\mathcal{U}(\tilde{\omega})^{\leq\lambda}]} \arrow[d] & \left(T\mathcal{U}(\tilde{\omega})^{\leq\lambda, \nabla_{\mathrm{naive}}}\right)^{p\text{-flat}} \arrow[l, hook'] \arrow[d] \arrow[dd, bend right] \\
{[\mathrm{Spf}\, R^{\leq\lambda,\tau}_\rho / G_n]} \arrow[r, "\sim"] \arrow[d, "\text{completion at a point}"] & {\left[T\mathcal{U}(\tilde{\omega})^{\leq\lambda,\nabla_{\mathrm{true}}} / T_{\mathrm{conj}}\right]^{p\text{-flat}}} \arrow[r, hook] \arrow[d, equal] & {[T\mathcal{U}(\tilde{\omega})^{\leq\lambda} / T_{\mathrm{conj}}]} \arrow[d, "-\cdot\tilde{\omega}_\tau"] & {\left[T\mathcal{U}(\tilde{\omega})^{\leq\lambda,\nabla_{W}} / T\right]^{p\text{-flat}}} \arrow[l, hook'] \\
 & X^{(\tilde{\omega})}_{n,\leq\lambda,\tau} & \mathcal{R}_n & \mathcal{U}(\tilde{\omega})^{\leq\lambda,\nabla_W} \arrow[d] \\
 & & & M(\lambda)^{\nabla_\tau}
\end{tikzcd}$$

$$T\mathcal{U}(\tilde{\omega})^{\leq\lambda, \nabla_{\mathrm{true}}} = \left\{ A \in L\mathfrak{g}(R) \;\middle|\; \underbrace{e\,\frac{v\,dA}{dv} A^{-1} + [s^{-1}t_\mu, A]A^{-1}}_{\text{has log poles}} + \mathcal{O}(p^N) \right\}$$

$$(\quad)^{\nabla_W} = \left\{ \underline{A \in L\mathfrak{g}(R) \;\middle|\; \left(e\,\frac{v\,d}{dv} A + [s^{-1}t_\mu, A]\right) A^{-1}} \right\}$$

has log poles

②

Note: Write $M(\leq\lambda)^{\nabla_{nv}} = M(\leq\lambda)^{\tau,\nabla_{nv}}$ if want to stress out the type

Strategy: $\left(TU(\tilde{\omega})^{\leq\lambda}\nabla_{nv}\right)^{p\text{-flat},\Lambda_p}$ has ≤ 1 irred component of maximal dim^n $= n + \dim B^G =$
$= \dim \tilde{X}_{n,\leq\lambda,\tau}$

$\downarrow$

[Study component of algebraic model " "]

Thm: ① $M(\leq\lambda)^{\nabla_{nv}}\left[\frac{1}{p}\right] = \bigsqcup P_{\lambda'}\backslash G$

$$\| $$

$$\left(M(\leq\lambda)^{\tau,\nabla_{nv}}\left[\tfrac{1}{p}\right]\right) \cap L^+\mathcal{G}(v+p)^{\lambda'}L^+\mathcal{G}$$

$M(\lambda')^{\nabla}$: closure of $P_{\lambda'}\backslash G$ in $\mathcal{G}r$

② $U(\tilde{\omega})^{\leq\lambda,\tau,\nabla}$ unibranch at $\tilde{\omega}$ i.e. $(U(\tilde{\omega})^{\lambda,\tau,\nabla})^{norm}$ $\ni p^t$

$$\|$$

$U(\tilde{\omega})^{\leq\lambda} \cap M(\lambda')^{\nabla_{nv},\tau}$ $\qquad\downarrow$ $\qquad U(\tilde{\omega}) \ni \tilde{\omega}$

Strategy: Define the above over $L\mathcal{G}, \dots, /\mathbb{A}^1_{\mathbb{Z}}$ moduli interpretation: moduli of $\mathcal{G}$-torsors P + trivialization over $\mathbb{A}^1$ away from $v=t$; meromorphic connection with log poles at $v=t$

$$M(\leq\lambda)^{\nabla}_{univ} = \left\{A \in L^+\mathcal{G}\backslash L\mathcal{G}, \left(\frac{v\,dA}{dv}\right)A^{-1} + [a, A]A^{-1} \text{ has log poles at } v=t\right\}$$

$U(\tilde{\omega})^{\leq\lambda,\nabla}$

$$\downarrow$$

$$\text{“}t\text{”} \to \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^n_{\mathbb{Z}} \leftarrow \text{“}a\text{”}$$

Prove the theorem for $M(\leq\lambda)^{\nabla}_{univ}$.

connection with residue a

[Away from $v=0$ ∇_a log at $v=0$, & residue a; Trivial bundle with such connection]

Away from $v=t$, we ask P to be the trivial bundle with pullback of the connection ∇_a, with log poles at $v=0$

③

The map is:

<u>Prove</u>: $M(\leq\lambda)^{\nabla}_{\mathrm{univ}} \cap L^+\mathcal{G}(v+t)^{\lambda'} L^+\mathcal{G}$

$$L^+\mathcal{G}((v+t)^{\lambda'}) L^+\mathcal{G} \to P_{\lambda'} \backslash GL_n$$

$$(v+t)^{\lambda'} GL_n \leftarrow \text{is a section}$$

Pf of ②:

$\boxed{\text{To show } M(\lambda)^{\nabla}_{\mathrm{univ}} \text{ is unibranch at } \tilde{\omega}, \text{ it suffices to show the base change to } \mathbb{A}^1_{\mathbb{F}_p} \times \mathbb{A}^{\omega} \text{ is}}$

$$(M(\lambda)^{\nabla}_{\mathrm{univ}})^{\mathrm{norm}} \to M(\lambda)^{\nabla}_{\mathrm{univ}}$$

$$(M(\lambda)^{\nabla}_{\mathrm{univ}})^{\mathrm{norm}} \times_{\mathbb{A}^1_{t,\mathbb{Z}}} \mathbb{A}^1_{t,\mathbb{F}_p}$$

$$\downarrow$$

$$(M(\lambda)^{\nabla}_{\mathrm{univ}}) \underline{\qquad}$$

norm → normalization

Key Fact: X irred. variety / k field, X has a <u>contracting</u> $\mathbb{G}_m$-action [i.e. extends to $\mathbb{A}^1$]

& $|X^{\mathbb{G}_m}| = pt$ then X is unibranch at x

Pf

$$\begin{array}{ccccc} (X^{\mathrm{norm}})^{\mathbb{G}_m} & \subseteq & X^{\mathrm{norm}} & \circlearrowleft & \mathbb{A}^1 \\ \downarrow & & \downarrow & & \\ x \in X^{\mathbb{G}_m} & \in & X & \circlearrowleft & \mathbb{A}^1 \end{array}$$

hence $X^{\mathrm{norm}} \twoheadrightarrow (X^{\mathrm{norm}})^{\mathbb{G}_m}$

$y \longmapsto \left(\lim_{t\to 0} t\cdot y\right)$

& X^{norm} is irred.

$\blacksquare$

4

$\mathcal{U}(\varpi)^{\lambda,\nabla} \subseteq \mathcal{U}(\varpi)$ & $L^+\mathcal{G} \backslash L\mathcal{G} \circlearrowleft T^{ext} = T \times \boxed{\mathbb{G}_m}$ dilatation of loop variable

$\downarrow \wr$ $\qquad\qquad$ $\downarrow$ $\qquad$ act on $L\mathcal{G}(t)$

$(L^{--}\mathcal{G})_{\varpi}$ $\qquad$ $\mathbb{A}^1_{\bar{\mathbb{F}}_p}$ $\qquad$ $= \left\{ X \in GL_n(R((v-t))) \;\middle|\; X \text{ upper } \nabla \bmod v \right\}$

& $R((v-t)) \simeq R((v-rt))$

$v \longmapsto r^{-1}v$

Key: find $\mathbb{G}_m \to T^{ext}$ s.t. it acts on $\mathcal{U}(\varpi)$, & contracting action here

e.g. $\varpi=1$, $\mathcal{U}(1) = L^-\mathcal{G} = \left\{ X \in GL_n(R[\frac{1}{v-t}]),\ X \bmod \frac{1}{v-t} \text{ is upper } \nabla,\ \bmod \frac{v}{v-t} \text{ is lower } \Delta \right\}$

$$\rightsquigarrow \begin{pmatrix} 1+\frac{c}{v-t} \cdots & \\ \frac{v}{v-t}\left(1+\frac{1}{v-t}\cdots\right) & \end{pmatrix} \underset{\rightsquigarrow}{\xrightarrow{v \mapsto r^{-N}v}} \begin{pmatrix} 1+\frac{c}{r^{-N}v-t} & \\ \frac{r^{-N}v}{r^{-N}v-t}\left(\cdots\right. & \end{pmatrix}$$

all weights are $\geq N$ except for those one below diagonal...

$$= \begin{pmatrix} 1+r^{N}\cdot\frac{c}{v-tr^{-N}}+\cdots & \\ & \end{pmatrix}$$

⑤

Finally $\mathcal{U}(\infty)^{\le,\nabla}$ is stable under $\mathbb{G}_m$-action by $v \mapsto r^{-1}v$

since "$\left(\frac{v\,d}{dv}A\right)A^{-1} + A\underline{a}A^{-1} \in \frac{1}{v-t}\cdot$ integral" is a $\mathbb{G}_m$-stable condition

Last step: deduce the result for $M(\lambda)^{\tau,\nabla}_{\mathbb{Q}_p}$ from $M(\lambda)^{\nabla}_{\mathrm{univ}}$.

(warning: normalisation does not commute with b/c!)

Prop: $\exists\ U \longrightarrow \mathbb{A}^n$ étale s.t. normalization commutes with b/c

$$\begin{tikzcd} M(\lambda)^{\nabla}_{\mathrm{univ},U} \arrow[d] \\ \mathbb{A}^1 \times U \end{tikzcd}$$

Finally: specialize to $t \mapsto -p$, $\underline{a} \mapsto s(\mu)$

Upshot: $\exists$ local model diagram

$$\begin{tikzcd} & \boxed{} \arrow[dl] \arrow[dr] & \\ \mathcal{X}_{\le\lambda,\tau} & & \left(M(\le\lambda)^{\tau,\nabla_\infty}\right)^{p\text{-flat}} \end{tikzcd}$$

not canonical, but it's canonical mod p

&

$$\begin{tikzcd} & & \left[\left(\tilde{M}(\le\lambda)^{\tau,\nabla_\infty}\right)^{p\text{-flat}}_{\mathbb{F}}\right] \arrow[dll] \arrow[dr] \arrow[rr, hook] & & \mathcal{F}\ell \arrow[d] \\ \left[M(\le\lambda)^{\tau,\nabla_\infty}/T\right]^{p\text{-flat}}_{\mathbb{F}} = \mathcal{X}_{\le\lambda,\tau} \arrow[r, hook] & \mathcal{R}_\infty & & \left[\left(M(\le\lambda)^{\nabla_\infty}\right)^{p\text{-flat}}_{\mathbb{F}}\right] & \mathcal{F}\ell \end{tikzcd}$$

$\rightsquigarrow (\mathcal{X}_{\leq\lambda,\tau})_{red} = [\tilde{M}(\leq\lambda)^{\tau,\nabla_{\infty}}_{red} / T]$

$\boxed{\lambda \text{ regular}}$

Thm: 1. Irr. cmpts. of $M(\leq\lambda)^{\tau,\nabla_{\infty}}_{\mathbb{F}}$ are given by closures of $M(\leq\lambda)^{\nabla_{\infty}}_{\mathbb{F}} \cap (I \tilde{\varpi} I)$, $\tilde{\varpi} \in \mathrm{Adm}^{reg}(\lambda)$ (call it $\mathcal{C}_{\tilde{\varpi},\leq\lambda,\tau}$)

2. $\exists$ canonical bijection

$$\mathcal{C}_{\tilde{\varpi},\leq\lambda,\tau} \leftrightarrow JH(W(\lambda-\eta)\otimes\sigma(\tau))$$

characterized by $z(\mathcal{C}_{\tilde{\varpi},\leq\lambda,\tau}) = \mathcal{X}_{red}(\sigma)$

component of $(\mathcal{X}_n)_{red}$ labelled by σ.

Corollary: $(\mathcal{X}_{\leq\lambda,\tau})_{red} = \bigcup_{\sigma\in JH(W(\lambda-\eta)\otimes\sigma(\tau))} \mathcal{X}_{red}(\sigma)$.

Rmk: this characterizes $\bar{\rho}$ which admit pcris lifts with HT $\leq\lambda$, type τ

Bao - 5

Previous lecture :

$$\begin{tikzcd}[column sep=small]
 & \prod_{\tilde{\omega}} T\mathcal{U}(\tilde{\omega})^{\leq\lambda,\tau,\nabla} \arrow[dl] \arrow[dr] & \\
\mathcal{X}_n^{cr,\leq\lambda,\tau} & & M(\lambda)^{\nabla,\tau} \hookrightarrow Gr
\end{tikzcd}$$

non canonical
but canonical
mod p

$$\left(\mathcal{X}_n^{cr,\leq\lambda,\tau}\right)_{\mathbb{F}} \xrightarrow{\sim} \left[\tilde{M}(\lambda)^{\nabla,\tau}/T\right]_{\mathbb{F}} \supseteq \mathrm{Adm}(\lambda)$$

$$\bar{\rho} \longleftrightarrow \tilde{\omega} \quad \text{iff } \bar{\rho} \text{ is semisimple}$$

&

$$\left(\tilde{M}(\lambda)^{\nabla,\tau}\right)^{\wedge}_{\tilde{\omega}} \quad \text{is irreducible}$$

Key: $\bar{\rho}^{ss}$ & suff. generic then $R_{\bar{\rho}}^{\eta,\tau}$ is a domain

$$\Rightarrow \left(\mathcal{X}^{cr,\leq\lambda,\tau}\right)_{red} = \bigcup_{\sigma\in JH\left(\overline{W(\lambda)\otimes\sigma^{cr}(\sigma)}\right)} \mathcal{X}^{\sigma}$$

(requires global input!)

(2)

Global setup:

G unitary group $/F^+$, F/F^+ unramified at p.

quasi split $\forall v$ (G/F)

split at $v|p$

$G(F^+_\infty)$ compact

$\bar{r}: G_F \to GL_n(\bar{\mathbb{F}}_p)$ automorphic w.r.t. G; $U = G(\mathcal{O}_{F^+,p}) \cdot U^p$

$\leftrightarrow \phi$; $\mathfrak{m} = \ker(\mathbb{T} \xrightarrow{\phi} \bar{\mathbb{F}}_p)$

U^p minimal w.r.t. $\bar{r}$ (tame level)

Functor: $\mathcal{O}[\prod_{v|p} GL_n(\mathcal{O}_{F^+_v})]\text{-mod} \xrightarrow{M} \mathbb{T}\text{-mod}$

$$V \longmapsto H^0(Y_U, V)_{\mathfrak{m}}$$

Geometrization of SWC: Describe $W(\bar{r}) = \{\sigma \text{ SW st. } M(\sigma) \neq 0\}$

Expectation (GHS) If $\bar{\rho}_v = \bar{r}|_{G_{F^+_v}}$ is ssimple, suff. generic then

$$W(\bar{r}) = \bigotimes W^{GHS}(\bar{\rho}_v)$$

Recipe for $W^{GHS}(\bar{\rho}_v) = \{ F(w_0 t_{-\eta} \cdot \lambda),\ F(\lambda) \in JH(R_s(\mu)) \}$

if $\bar{\rho}_v = \tau(s, \mu)$

Rmk: $\# W^{GHS}(\bar{\rho}_v) \to$

n	2	3	4	5
	2	9	88	>1000

③

No conjecture for $\bar\rho$ <u>not</u> semisimple, but crystalline lift conjecture:

Turns out to be FALSE!

$\boxed{F(\lambda) \in W(\bar\rho_v) \text{ if } \bar\rho_v \text{ has a crystalline lift of weight } \lambda.}$

Global setup has many patching functors M_∞:

$$M_\infty : \text{Fin. gen. } GL_n(\mathcal{O}^+_{F,p})\text{-mod} \longrightarrow \text{Modules} / \bigotimes_{v|p} R_{\bar\rho_v}[[x_1, \dots, x_h]]$$

s.t. 1. Recover M from M_∞: $M_\infty(V) \neq 0$ iff $M(V) \neq 0$

2. $M_\infty(W(\lambda-\eta) \otimes \sigma^{cr}(\tau))$ supported on $\overbrace{\bigotimes_{v|p} R^{\lambda,\tau}_{\bar\rho_v}}^{d\text{-dim'l}}[[x \dots]]$

& maximal CM over its support → cycle $Z^{cr,\lambda,\tau}_{out, M_\infty}$ in $\bigotimes R^{\lambda,\tau}_{\bar\rho_v}/p$

$\hookrightarrow \subseteq \mathcal{X}_d$

cycle of irr cmpts of $\bigotimes_{v|p} R^{\lambda_v,\tau_v}_{\bar\rho_v}$

3. $M_\infty(\sigma)$ maximal CM, supp. is equidim'l of dim $d-1$

ConjB: $\exists$ cycles $Z_\sigma \subset \mathcal{X}_d$ s.t.

1. $\forall$ global setup, $\underset{\text{supp } M_\infty(\sigma)}{\underset{\shortparallel}{Z_{out}^{\sigma}}} = \sum n^{cr}_\sigma(\lambda,\tau) \cdot Z_\sigma^{\bar\rho_v}$

$n^{cr}_\sigma(\lambda,\tau) \updownarrow [W(\lambda-\eta) \otimes \sigma(\tau) : \sigma]$

2. BMC: $\mathcal{X}_{\mathbb{F}}^{cr,\lambda,\tau} = \sum n^{cr}_\sigma(\lambda,\tau) \cdot Z_\sigma$

<u>Rmk</u>: 1, 2 overdetermine Z_σ

If Conj. B: $Z_\sigma \geq 0$

$\sigma \in W(\bar{r}) \iff \bar{\rho}_\sigma \in Z_\sigma$

Thm (LLLM): ① $\forall \sigma$ suff gen., $\exists Z_\sigma \subseteq \mathcal{X}_n$ s.t. conj. B① holds for $\bar{\rho}_\sigma$ ss

② Conj. B② holds for all (λ, τ) suff. generic.

Rem: ① $\mathcal{X}^\sigma \subseteq Z_\sigma$ with mult. 1

$\sigma' \uparrow \sigma$

$\Downarrow$

$$Z_\sigma = \mathcal{X}^\sigma + \sum_{\sigma' \text{ covered by } \sigma} a(\sigma,\sigma') \mathcal{X}^{\sigma'}$$

σ covers σ' $\iff$ $\left[\begin{array}{l}\forall \text{ tame } \tau, \\ \sigma \in JH(\bar{\tau}) \\ \text{then do } \sigma' \in JH(\bar{\tau})\end{array}\right]$

For $n=3$, $Z_\sigma = \mathcal{X}^\sigma$ irreducible

$n \geq 4$ seems fake !

② $W^{BM}(\bar{\rho}_v) := \{\sigma_v,\ \bar{\rho}_v \in Z_{\sigma_v}\} \subseteq W^{GHS}(\bar{\rho}_v^{ss})$

For $\bar{\rho}_v^{ss}$, we conjecture the equality

$\Uparrow$

$[\bar{\rho}^{ss} \in \mathcal{X}^\sigma \text{ iff } \sigma \in W^{GHS}(\bar{\rho}^{ss})]$

($\Rightarrow$ known)

③ For $n=3$, you know $\hat{Z}^{\sigma}{}^{\bar{\rho}_v} = Z^\sigma_{out}$ even for $\bar{\rho}$ not ss simple.

Sketch of pf:

fix a choice of presentation: $[\sigma] = \sum_{\substack{\tau \\ \text{generic}}} n_\sigma(\tau)[\tau] + (\text{non generic stuff})$

Define $Z_\sigma := \sum n_\sigma(\tau) \cdot \mathcal{X}^{\eta,\tau}$ (no reason a priori why indep. on pres. & globalization)

$\forall$ global setup: $Z^\sigma_{out} = \sum n_\sigma(\tau) \cdot Z^{\eta,\tau}_{out}$ by exactness of M_∞

A priori, $Z^{\eta,\tau}_{out} \subseteq (\mathcal{X}^{\eta,\tau})^{\wedge \bar{\rho}_v}$ but "=" holds since $M_\infty(\sigma^{cr}(\tau))$ has full support.

(Key: $\bar{\rho}^{ss} \Rightarrow R^{\eta,\tau}_{\bar{\rho}^{ss}}$ irreducible)

This implies: $Z_\sigma^{\wedge \bar{\rho}_v} = Z^\sigma_{out}$; in particular, Z^σ_{out} is purely local

$\Rightarrow$ the collection Z_σ obey the BM equations. $\Rightarrow$ get (1).

b/c $\mathcal{X}^\sigma \leftrightarrow C_\sigma \circlearrowleft T$ of some $M(\lambda)^{\tau,\nabla}$ $\subseteq$ Fl

T-action on proper, $\neq \emptyset$ variety has fixed pt.

For (2), if $\bigoplus_{\sigma \text{ generic}} \mathbb{Z} \cdot \mathcal{X}^\sigma \hookrightarrow \bigoplus_{\bar{\rho}^{ss}} \bigoplus_\sigma \mathbb{Z}(\mathcal{X}^\sigma)^{\wedge \bar{\rho}^{ss}}$

$C \longmapsto \bigoplus C^\wedge_{\bar{\rho}^{ss}}$

Here, all BM eq^n hold for Z^σ

Remk: by partial order on σ' (covering), $\exists$ algorithmic way to compute $Z^\sigma = \mathcal{X}^\sigma + \sum a(\sigma,\sigma')\mathcal{X}^{\sigma'}$