**Abstract:**

BPS invariants are fundamental enumerative invariants for sheaves on Calabi-Yau threefolds, which determine other enumerative invariants of interest, such as Donaldson-Thomas or Gromov-Witten invariants. In these lectures, we discuss cohomological and categorical refinements of BPS invariants.

The plan for each lecture is is as follows:

**Lecture 1:**

We discuss quivers with potentials, which can be used to model the moduli of semistable sheaves on a Calabi-Yau threefold. We then define BPS cohomology for quivers with potential and discuss a theorem of Davison-Meinhardt which says that the vanishing cohomology of the stack of representations of a quiver with potential is generated by the BPS cohomology. We mention the computation of BPS for points on the affine three dimensional space and discuss an example of the Davison-Meinhardt theorem. Finally, time permitting, we discuss dimensional reduction, which relates, in some cases, vanishing cohomology for a regular function on a smooth stack with Borel-Moore homology of a (possible singular) stack.

**Lecture 2:**

One of the main ingredients in the study of categorifications of BPS and Donaldson-Thomas invariants is the theory of window categories. For a smooth quotient stack and a stability condition, there are window categories (defined by Segal, Halpern-Leistner, Ballard-Favero-Katzarkov) inside the category of coherent sheaves on the stack which are equivalent to the category of coherent sheaves on the semistable stack. Further, one can decompose the category of the stack in a window category and subcategories supported on the unstable loci. We will discuss an example of these theorems.
Next, we discuss a construction of Spenko-Van den Bergh of noncommutative resolutions of singularities of coarse spaces of smooth quotient stacks. We compare their definition with window categories. The construction of Spenko-Van den Bergh provides a categorical version of the BPS cohomology of a quiver with zero potential.

**Lecture 3:**

In this 3rd lecture, we review matrix factorizations associated with regular functions on smooth quotient stacks, and explain that they are regarded as categorifications of vanishing cycle cohomologies. In special cases, they are equivalent to derived categories of somequasi-smooth derived stacks, called Koszul equivalence. We then introduce quasi-BPS categories for quivers with potential, and also for preprojective algebras via Koszul equivalence. We explain that they are building blocks of the matrix factorization categories of moduli stacks of representations of the quivers, and derived categories of derived moduli stacks of representations of preprojective algebras, via semiorthogonal decompositions induced by categorical Hall products.

**Lecture 4:**

The topic of this 4th lecture is two hold: (I) We focus on the tripled loop quiver with potential and its quasi-BPS categories. By considering the framing, we show that there is a semiorthogonal decomposition of the DT category for the Hilbert scheme of points on C^3 in terms of categorical Hall products of quasi-BPS categories. We explain its relationship with wall-crossing formula of DT invariants and McKay correspondence (II) We construct quasi-BPS categories for K3 surfaces and the semiorthogonal decomposition of derived categories of derived moduli stacks of semistable objects on K3 surfaces into categorical Hall products of quasi-BPS categories. When the Mukai vector and the weight is coprime, we show that the reduced quasi-BPS categories are smooth, proper with etale locally trivial Serre functor, thus giving a twisted analogue of categorical crepant resolution of the singular symplectic moduli space of semistable objects on K3 surfaces.