HAUSDORFF SCHOOL
FOR ADVANCED STUDIES
IN MATHEMATICS

UNIVERSITÄT BONN

**Hausdorff School**

# "Circle Method Summer School"

**May 10 to 14 and May 17 to 21, 2021**

organized by
**Lillian Pierce, Oscar Marmon**

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## Abstracts

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**Kirsti Biggs** (University of Gothenburg)

### Ellipsephic applications of the circle method

**Abstract:** Ellipsephic sets are subsets of the natural numbers defined by digital restrictions in a given base — such sets have a fractal-like structure which can be seen as a p-adic analogue of generalised real Cantor sets. The recent work of Maynard on primes with missing digits can be seen as an ellipsephic problem, although in this talk we focus on smaller sets of permitted digits, one motivating example being the set of natural numbers whose digits are squares. I will discuss applications of the circle method to Diophantine problems involving ellipsephic sets, and highlight the key features of such results.

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**Julia Brandes** (Universit of Gothenburg)

### Simultaneous diagonal and non-diagonal equations

**Abstract:** The circle method can be used to effectively count solutions to diagonal equations. It can also be used (though less effectively) to count solutions to equations of general shape. Can we do both at once, without abandoning the special shape of the diagonal equations? We will outline a strategy to address this problem by combining approaches in the style of Birch's theorem with those stemming from systems of diagonal equations.

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**Timothy Browning** (IST Austria)

### The circle method over function fields and applications to geometry

**Abstract:** The solubility of cubic forms over arithmetically interesting fields has long been at the centre of number theory. I will briefly survey what is known for finite fields, local fields and global fields. The main goal is then to introduce the mechanics of the Hardy-Littlewood circle method over the function field $F_q(t)$ through the prism of cubic forms. After introducing the basics, I will explain how to deal with $F_q(t)$-solubility for diagonal cubic forms that are defined over the finite field $F_q$. It turns out that these techniques can also shed light on the geometry of the moduli space parameterising rational curves on varieties. I will illustrate this by showing how the $F_q(t)$-version of the circle method can recover classical facts about the smoothness/dimension/irreducibility of the Fano variety of lines on diagonal cubic hypersurfaces.

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**Jayce Getz** (Duke University)

### New avenues for the circle method

**Abstract:** Motivated by research arising from automorphic representation theory, I will present some ideas that should open up new avenues of research in the circle method. In the first half of the lectures I will discuss an adelic version of the delta-method of Duke, Friedlander, Iwaniec and Heath-Brown and state a (mostly conjectural) nonabelian analogue that I believe warrants further study. In the second half of the lectures I will discuss Poisson summation formulae and Fourier transforms for special families of varieties including, for example, the zero locus of a quadratic form. My hope is that they will allow the standard techniques of analytic number theory that rely on Fourier theory on a vector space to be broadly generalized.

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**Yu-Ru Liu** (University of Waterloo)

### An introduction to the circle method

**Abstract:** This course will focus on the basic principles of the circle method. We will start with Waring's problem about representations of positive integers as a sum of fixed powers. Then we will study Vinogradov's mean value theorem about a system of equal sums of powers. If time permits, we will consider extensions of these problems to the multidimensional setting and the function field.

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**Oscar Marmon** (Lund University)

### Rational points on quartic hypersurfaces

**Abstract:** By work of Heath-Brown and Hooley, it is known that the Hasse principle holds for non-singular cubic forms in at least nine variables. The situation for forms of higher degree is much less satisfactory. Browning and Heath-Brown established the Hasse principle for non-singular quartic forms in at least 41 variables, and Hanselmann subsequently showed that 40 variables suffice. In joint work with Pankaj Vishe, we have been able to drastically reduce the number of variables needed to 30. To obtain the improvement, we combine Heath-Brown's delta-symbol version of the circle method with a van der Corput differencing technique.

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**Ritabrata Munshi** (Indian Statistical Institute)

### The circle method and the analytic theory of L-functions

**Abstract:** The aim of this series of talks will be to introduce variants of the delta method and to show how they can be employed to tackle various problems in the analytic theory of automorphic forms and L-functions. We will start by applying the classical Kloosterman's circle method and/or the delta method of Duke, Friedlander and Iwaniec to prove t-aspect subconvex bounds for degree two and three L-functions. We will also look at the subconvexity problem in the twist aspect, and in particular prove the Burgess bound for degree two L-functions. (A level lowering technique plays a crucial role in this delta symbol approach to subconvexity.) Next we will introduce the Bessel

delta method which is tailor-made to tackle problems related to GL(2) Fourier coefficients. As an application we will study exponential sums involving such Fourier coefficients and prove a sub-Weyl bound in this context. Finally we will consider delta methods which are derived from Petersson trace formula or Kuznetsov trace formula, and see some applications to spectral/weight aspect subconvexity problems.

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**Simon Myerson** (University of Warwick)

### Repulsion: a how-to guide

**Abstract:** Consider the integral zeroes of one or more, not necessarily diagonal, integral polynomials in many variables with the same degree. The basic principles for applying the circle method here were laid out by Birch. One way to improve on his work is repulsion: showing that the exponential sum over the polynomials can be large only on small, well separated regions. Unusually for improvements on Birch's work this idea has been successfully applied to systems which are not particularly singular and which contain many polynomials. To begin I will ask: what about Birch's work suggests that repulsion could be an improvement? I will then discuss the quadratic and higher degree case in detail, and an application to systems of forms with real coefficients.

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**Lillian Pierce** (Duke University)

### Applications of the circle method in harmonic analysis

**Abstract:** Around 1990, Bourgain realized that the circle method was an effective tool to understand the behavior of certain discrete operators in harmonic analysis, with interesting applications in ergodic theory. This opened the door to studying a wide range of discrete analogues of operators in harmonic analysis, by combining ideas of a dissection into major/minor arcs with the existing understanding of the associated real-variable operators. In this lecture series, we will introduce several exemplar problems of this type, provide some of the relevant analytic background, and demonstrate how ideas from the circle method play a role.

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**Damaris Schindler** (University of Göttingen)

### Beyond the circle method

**Abstract:** The last two weeks we have seen how important and useful the circle method is in counting integer solutions to systems of Diophantine equations. From a more geometric point of view the circle method provides us a valuable tool for counting rational points of bounded height on certain varieties. But what happens in situations where we don't have enough variables to apply the circle method? Or where we are not counting points on varieties but points close to manifolds? What if the height function on a projective variety does not directly lead us to a counting question in boxes, as is the situation where the circle method is best applied? These and related questions are going to be the topic of this talk. We are going to focus on situations where analytic tools play a key role in finding answers.

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**Pankaj Vishe** (Durham University)

**On the Hasse principle for complete intersections**

**Abstract:** Let $X \subset \mathbf{P}_{\mathbf{Q}}^{n-1}$ be a smooth projective complete intersection variety defined by a system of two cubic polynomials. We prove that $X$ satisfies the Hasse principle as long as $n \geq 40$. The key ingredient here is the development of a Kloosterman refinement for complete intersections over $\mathbf{Q}$. This is a joint work with Matthew Northey.

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