Courses Hausdorff School

TQFTs and their connections to representation theory and mathematical physics

Adrien Brochier (Paris): Factorization homology and representation theory

Abstract: The notion of ribbon category is the most fundamental example of the general dictionary between representation theory and low-dimensional topology: in one direction those structures provides well-behaved invariants of braids and links and, under additional assumptions, of 3-manifolds. In the other direction, they provide a "graphical calculus" for various structures arising in representation theory, and in particular in quantum algebra.

Factorization homology on the other hand, is a very general and arguably fairly abstract implementation of a similar kind of dictionary between certain "higher algebraic" structure and certain well-behaved invariants of topological n-manifolds.

It turns out that ribbon categories are examples of (and it some sense the ancestors of) the kind of higher algebraic structure that one can use as "coefficients" for the 2-dimensional version of factorization homology. The general formalism thus produces out of a ribbon category A, a machine that canonically attaches a category to any compact oriented surface.

The main goal of this lecture series will be to properly define factorization homology in that particular setting, to explain how it can be computed explicitly and to describe some applications of that formalism in quantum algebra and low-dimensional topology. On the one hand, this construction should be thought of as a refinement of Walker's notion of "skein category", and as such are closely related to the theory of skein modules of 3-manifolds. On the other hand, in the important case where A is the category of representations of a so-called quantum groups, the category attached to a surface S leads to a canonical quantization of a certain Poisson structure on the character variety on S, discovered by Atiyah-Bott in the framework of 2d Yang-Mills theory.

As a warm-up we'll start the lectures with a very down to earth overview of classical character varieties and their diagrammatic description in terms of loops on surface (the "Wilson loops observables" of Yang-Mills theory). I'll emphasize algebraic aspects of that theory which also have a clear geometric interpretation in that particular case in a way which, hopefully, should help motivate various features that factorization homology ought to have.

We'll then spend some time thinking about "categorified linear algebra", a formalism which is necessary to define factorization homology of ribbon categories and to formulate its compatibility with gluing of surfaces along their boundaries. Along the way, I'll define one of our main example of

ribbon category, namely the category of representation of the quantum group associated with a complex reductive algebraic group.
Finally, I'll explain how all of this can be computed explicitly, the key ingredient being the so-called braided dual of the ribbon category A, a certain canonical algebra in A which shows up fairly often in quantum topology. Time permitting I'll explain some applications.

Alexej Davydov (Ohio): E_n algebras and deformations of tensor categories

Abstract: Lecture 1: moduli of tensor categories and tensor functors

motivation: moduli of associative algebras
groupoid of twisted forms of a monoidal functor
2-groupoid of twisted forms of a monoidal category
cosimplicial complex of endomorphisms of a monoidal functor
examples: modules over bialgebra; bimodules
change of scalars of tensor category

Lecture 2: deformation cohomology of tensor categories and tensor functors

motivation: deformation cohomology of associative algebras
tangent twisted forms of tensor categories and tensor functors
cochain complex of endomorphisms of a monoidal functor
examples: modules over bialgebra; bimodules
cup-product

Lecture 3: deformation cohomology and extensions

motivation: Hochschild cohomology of associative algebras
multi-category of functors between monoidal categories
deformation complex as Hochschild complex
relative extensions of bimodules
cup-product as Yoneda product
Hodge decomposition for symmetric functors

Lecture 4: E_n-structure on deformation cohomology

motivation: deformation cohomology of associative algebras
e_n-structure on self-extensions of the unit object
cosimplicial monoids and lattice path operads
examples: representations of Lie algebras

Lecture 5: quantum groups of type A

free symmetric category
Hecke categories
fibre functors and r-matrices

Tudor Dimofte (Edinburgh): VOA's and 3d TQFT's from supersymmetric QFT's

Abstract: These lectures will be an introduction to recent developments in understanding the mathematical structure of topological twists of 3d supersymmetric gauge theories, especially through the perspective of boundary vertex operator algebras (VOA's). The tentative plan is as follows.

Lecture 1 will be a review of an archetype for many of the modern/recent constructions: Chern-Simons theory with compact gauge group, and the WZW VOA on its boundary, developed in the late 80's and early 90's. I'll discuss line operators, state spaces, and 3-manifold invariants from physical and mathematical perspectives, and relate them to VOA constructions.

Lecture 2 will be a lightning review of some relevant VOA concepts: Heisenberg, Kac-Moody, and WZW algebras, free field algebras and their extensions, embeddings of VOA's in free fields, and a bit of representation theory. Some students have asked about preparation for this lecture course. Chapters 1-5 of Vertex Algebras and Algebraic Curves by Frenkel and Ben Zvi would be excellent introductory reading.

Lecture 3 will be an overview of 3d N=4 supersymmetric gauge theories: their defining data and fields, Higgs and Coulomb branches of vacua, and the duality known as 3d mirror symmetry. I also hope to begin discussing their topological twists.

Lecture 4 will continue the discussion of topological twists of 3d N=4 theories, with an identification of various mathematical objects in the twists: boundary vertex algebras, braided tensor categories of line operators, state spaces, and (heuristically at this point) 3-manifold invariants. I'll explain the action of 3d mirror symmetry on boundary VOA's for abelian theories – the nonabelian analogue is still unknown. I hope to present a conjecture (based on work with T. Creutzig and W. Niu) on how quantum groups appear in abelian 3d N=4 theories as well.

Lecture 5 will be about deformations of the putative 3d TQFT's arising from twists of 3d N=4 gauge theories, focusing on deformations by auxiliary flat connections. I'll explain how this connects many constructions that have appeared in mathematics and mathematical physics – including Kashaev-Reshetikhin/Costantino-Geer-Patureau-Mirand invariants (of 3-manifolds endowed with flat connections), Reidemeister-Ray-Singer torsion (with a background connection), and vertex algebras with large center. I'll also attempt to indicate how such deformations fit into the physical realization of the geometric Langlands program.

Mikhail Kapranov (Kavli IPMU): Higher Segal spaces

Abstract: This series of 5 lectures will give an introduction to the study of spaces (more precisely, of simplicial objects of model categories) satisfying

"higher" analogs of the classical Segal condition. The classical condition (1-Segal, in our terminology), expressing a simplicial object in terms of its 0- and 1-simplices, is at the basis of one of the approaches to the theory of $(\infty, 1)$-categories. So our d-Segal spaces for $d \geq 2$ have a marked flavor of "higher dimensional algebra" and are somewhat analogous (but not identical) to d-dimensional TQFTs. Most immediate is the case $d = 2$ when the 2-Segal property serves as a tool of establishing associativity of various versions and categorifications of Hall algebras. Most of the time will be devoted to this case.

The first, motivational lecture will discuss the classical notion of Hall algebras, a conceptual reason for their associativity and the definition of the Waldhausen S-construction whose properties underlie this reason. The second lecture will introduce the 2-Segal property holding for the S-construction and illustrate its use in the definition of Cohomological Hall Algebras in various dimension. The third lecture will introduce 2-Segal objects in general model categories and give a sufficient criterion for being 2-Segal in a way which relate the 1-Segal and 2-Segal properties. The fourth lecture will discuss various examples of 2-Segal spaces and the fifth one will survey some applications and generalizations ($d > 2$).

Most of the material will be based on joint work with T. Dyckerhoff.