Workshop

“"SPP meets TP": Variational methods for complex phenomena in solids”

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organized by
Georg Dolzmann (Regensburg), Xavier Lamy (Toulouse), Angkana Rüland (Heidelberg)

Abstracts

Amit Acharya (Carnegie Mellon University)

An action functional for nonlinear dislocation dynamics

Abstract: Dislocations are the physical defects whose motion and interaction are responsible for the plasticity of crystalline solids. The physics can be characterized by a system of nonlinear PDE which does not naturally emanate from a variational principle. We describe the development of a family of dual variational principles for this primal system with the property that the Euler-Lagrange system of each of its members is the primal system in a well-defined sense. We illustrate the main idea of the scheme and its viability by applying it to compute approximate solutions to the linear heat, and first-order, scalar wave equations, and 1-d, nonconvex elastostatics.

Jonathan Fabiszisky (University of Münster)

Multiscale structures in compliance minimization

Abstract: As a model problem for variational pattern formation in materials science we consider an optimal design problem, i.e. we aim to optimize the topology and geometry of an elastic material in a two-dimensional square domain. We minimize a weighted sum of the volume and compliance (the work done by the load) of the material distribution under a prescribed boundary load and study the regularizing effect of an additional weighted perimeter term. While the optimal pattern for uniaxial and shear-type boundary loads is already partly understood, the case of a hydrostatic boundary load is completely open. As a candidate for a nearly optimal structure we combine known optimal structures for circular domains with a circle packing similar to a so-called Apollonian gasket. To obtain information about the optimality of this construction, we investigate the necessity of the occurrence of microstructures in optimal structures.

Maria Stella Gelli (University of Pisa)

Asymptotic analysis of convolution-type energies on periodically perforated domains

Abstract: Starting from the pioneering work on fractional Sobolev norms by Bourgain, Brezis and Mironescu, variational limits of non-local functionals of convolution-type have been recently widely studied. As an application, continuous energies depending on finite differences appear in a natural way in the study of models of inhomogeneous media with an underlying periodic microstructure. Here we push forward the variational analysis by imposing a Dirichlet condition on a periodic perforation of the domain. The interplay among the period, the size of the perforations, and the approximation parameter affects deeply the asymptotic behaviour of the energies, highlighting different phenomena. In particular, we detect a regime with the appearence of a nonlocal capacity term in the variational limit. This is an ongoing project with R. Alicandro and C. Leone.

Janusz Ginster (HU Berlin)

Scaling Laws and the Emergence of Complex Patterns in Helimagnetism

Abstract: In this talk we discuss a variational approach to study pattern formation in Helimagnets. After a brief introduction to the technique of scaling laws, we turn our attention to studying the emergence of patterns in magnetic compounds near the helimagnetic/ferromagnetic transition point in case of incompatible Dirichlet boundary conditions for the spin field. The energy under investigation is a continuum approximation of a J1-J3-model. It contains three parameters, the first measuring the incompatibility of the boundary conditions, the second measuring the cost of changes between different chiralities and the third depends on different optimal angle velocities of the spin field in rows and columns. We prove a scaling law for the minimal energy in terms of these parameters. The constructions from the upper bound indicate that in some regimes branching-type patterns form close to the boundary of the sample. Eventually, we discuss how some of the results in the continuous setting can be transferred to the discrete J1-J3-model. – Joint work with Melanie Koser and Barbara Zwicknagl (both Humboldt-Universität zu Berlin, Germany).

Dorothee Knees (University of Kassel)

Analysis and simulation of a rate-independent phase-field damage model

Abstract: Within this talk, the focus is on rate-independent damage models. Since the corresponding phase-field energies in general are non-convex, we are faced with a discontinuous evolution of the phase-field variable. Solution concepts have to be carefully chosen in order to predict discontinuities that are physically reasonable. We focus here on the concept of balanced viscosity solutions and develop a convergence scheme that combines alternate minimization with a local minimization ansatz due to Mielke/Efendiev, [EM06]. We proof the convergence of the incremental solutions to balanced viscosity solutions and illustrate the behaviour of the numerical scheme with some examples, [BRKM22]. [EM06]

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Marta Lewicka (University of Pittsburgh)

Geodesics and isometric immersions in kirigami

Abstract: Kirigami is the art of cutting paper to make it articulated and deployable, allowing for it to be shaped into complex two and three-dimensional geometries. We are concerned with two questions: (i) What is the shortest path between points at which forces are applied? (ii) What is the nature of the ultimate shape of the sheet when it is strongly stretched? Mathematically, these questions are related to the nature and form of geodesics in the Euclidean plane with linear obstructions (cuts), and the nature and form of isometric immersions of the sheet with cuts when it can be folded on itself. In particular, we prove that the full family of polygonal geodesics can be simultaneously rectified into a straight segment via a piecewise affine isometry. Join work with Qing Han and L. Mahadevan

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Peter Lewintan (University of Duisburg-Essen)

L^1-Korn-Maxwell-Sobolev inequalities in all dimensions

Abstract: We characterize all linear part maps $\mathcal{A}[\cdot]$ (e.g. $\mathcal{A} = \text{sym}$) which may appear on the right hand side of Korn-Maxwell-Sobolev inequalities for incompatible tensor fields P. The correction term $\operatorname{Curl} P$ appears thereby in the L^1 norm on the right hand side. Different from previous contributions, the results to be presented are applicable to all dimensions and optimal. This particularly necessitates the distinction of different constellations between the ellipticities of $\mathcal{A}$ and the underlying space dimensions n, especially requiring a finer analysis in the two-dimensional situation. These results are based on a joint work with Franz Gmeineder (Konstanz) and Patrizio Neff (Essen).

Cy Maor (Hebrew University of Jerusalem)

From Volterra's dislocations to strain-gradient plasticity

Abstract: Dislocations can be viewed as stress-free elastic bodies that undergo some "cut and weld" procedures (corresponding to a glide mechanism in a crystal); they were first studied by Volterra in the early 20th century, and were related to plasticity since the 1930s. However, only in the last 15 years plasticity models were rigorously derived as Γ-limits of models of finitely many dislocations. In most of these works, the single dislocation is modeled as a strain field whose curl is a delta function representing the Burgers vector; this is, in a sense, a pre-assumed linearization of Volterra's model. In this talk I will describe Volterra's model, and show how strain gradient plasticity in two-dimensions can be obtained directly from it. No previous knowledge in dislocation theory or plasticity will be assumed. Joint work with Raz Kupferman.

Alice Marveggio (Institute of Science and Technology Austria)

Approximation of multiphase mean curvature flow by means of the Allen-Cahn phase-field model

Abstract: The evolution of interfaces by mean curvature flow features the occurrence of topology changes and geometric singularities. An alternative approach for the description of the underlying dynamics is that of phase-field models of Allen-Cahn type, which replace sharp interfaces with diffuse interface layers of width of order $\epsilon > 0$. In the limit of vanishing interface thickness ($\epsilon \to 0$), the consistency of phase-fields models with their sharp-interface models has to be proven. In this talk, we give a rigorous proof for (unconditional) convergence of the vectorial Allen-Cahn equation with a suitable potential with $N \geq 3$ distinct minima towards multiphase mean curvature flow, assuming that a classical (smooth) solution to the latter exists. Our result holds in two and three ambient dimensions and starting from well-prepared initial data. We even establish a rate of convergence. Our approach relies on a suitable notion of relative energy for the vectorial Allen-Cahn equation with multi-well potential, in particular on the recent concept of "gradient flow calibrations" for multiphase mean curvature flow. This is based on joint work with J. Fischer.

Alexander Mielke (WIAS Berlin and Humboldt-Universität zu Berlin)

Viscoelastic fluid models for geodynamic processes in the lithosphere

Abstract: We study geodynamical flow describing the slow evolution of crustal faults between different lithospheric plates. The rock material can be treated as a visco-elasto-plastic fluid when the velocities are measured in the subcentimeter range per year. The continental drifts can be seen as externally described displacement conditions on the boundary.
In the full model studied in [1] we consider a Navier-Stokes equation where the stress tensor consist of a Newtonian part and an internal part that has its own dissipative solution. For this model global existense of weak solutions can be shown.
The simplified model in [2] only considers plane shear motion and thus can be analyzed in more detail. We study (non-equilibrium) steady state of localized slip zones and provide numerical simulation showing periodic effects where sticking phases alternate with slipping phases quite similar to the so-called "episodic tremor and slip".

This research is joint work with Thomas Eiter, Katharina Hopf, and Tomáš Roubíček and was partially supported by the DFG via the SFB 1114 and SPP 2256.

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Maria Giovanna Mora (University of Pavia)

Explicit minimizers for anisotropic Coulomb energies

Abstract: Nonlocal interaction energies play a pivotal role in describing the behavior of large systems of particles, in a variety of applications. Traditionally, the focus of the mathematical literature on nonlocal energies has been on radially symmetric potentials, which model interactions depending on the mutual distance between particles. The mathematical study of anisotropic potentials, despite their natural occurrence in modeling interactions where a preferred direction of interaction is present, has on the other hand been very limited until recently. In this talk we will consider a general class of anisotropic energies of Coulomb type in three dimensions and give a complete characterization of their minimizers, under the sole assumption of non-negativity for the Fourier transform of the interaction kernel.

Gokul Nair (Cornell University)

Energy Minimizing Configurations for Single-Director Cosserat Shells

Abstract: We consider a class of single-director Cosserat shell models accounting for both curvature and finite mid-plane strains. We assume a polyconvexity condition for the stored-energy function that reduces to a physically correct membrane model in the absence of bending. Among other things, we argue that this model is suitable for predicting wrinkling. With appropriate growth conditions, we establish the existence of energy minimizers. The local orientation of a minimizing configuration is maintained via the blowup of the stored energy as a version of the local volume ratio approaches zero.

Filip Rindler (University of Warwick)

Elasto-plasticity driven by dislocation movement

Abstract: This talk presents some recent progress for models coupling large-strain, geometrically nonlinear elasto-plasticity with the movement of dislocations. In particular, a new geometric language is introduced that yields a natural mathematical framework for dislocation evolutions. In this approach, the fundamental notion is that of 2-dimensional "slip trajectories" in space-time (realized as integral 2-currents) and the dislocations at a given time are recovered via slicing. This modelling approach allows one to prove the existence of solutions to an evolutionary system describing a crystal undergoing large-strain elasto-plastic deformations, where the plastic part of the deformation is driven directly by the movement of dislocations. This is joint work with T. Hudson (Warwick).

Mohammad Sarhil (University of Duisburg-Essen)

Aspects on modeling the size-effects of mechanical metamaterials via the relaxed micromorphic model

Abstract: Metamaterials are attracting growing attention lately. They can be tailored to meet a special application where their properties are determined mainly by their specific sophisticated engineered structures. However, they often exhibit size-effects which cannot be modeled by the standard Cauchy theory or the classical first-order homogenization procedures. Therefore, other generalized continuum theories can be employed such as Cosserat, second-gradient, or micromorphic theories.
The relaxed micromorphic model [1,2] successfully captures the main microscopic and macroscopic mechanical properties of the metamaterials for many applications, for example [3]. It avoids the complexity of the general micromorphic theory by using fewer material parameters and exhibits bounded stiffness for small specimens. The relaxed micromorphic model employs the Curl of a micro-distortion field, similar to the Cosserat model, but with the full kinematics of the micromorphic theory. Therefore, the solution exists in $H(curl, B)$ for the micro-distortion field while the displacement is in $H^1(B)$.
In our talk, we present our recent developments and findings regarding using the relaxed micromorphic model to model the size-effects of assumed 2D metamaterial beams subjected to bending, [4,5]. The finite element formulation, parameters identification, boundary conditions and homogenization of the relaxed micromorphic model will be discussed.
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Bernd Schmidt (Universität Augsburg)

Fracture and Curvature in Thin Structures

Abstract: We report on some recent results on effective theories for thin and ultrathin structures which are subject to both elastic deformations and fracture. In particular, we will consider the formation of folds and cracks in two scenarios: 1. atomistically thin rods and 2. membranes and plates with soft inclusions.

Antonio Tribuzio (Heidelberg University)

Energy scaling of singular-perturbation models involving higher-order laminates

Abstract: Motivated by the appearance of complex microstructures in the modelling of shape-memory alloys, we study the energy scaling behaviour of some N-well problems with surface energy given by a singular higher-order term. In the case of absence of gauge invariances (e.g. with respect to the action of SO(n) or Skew(n)), we provide an ansatz-free lower bound which relies on a bootstrap argument in Fourier space and gives evidence of the higher order of lamination involved. The upper bound is provided by iterated branching constructions. In the end, we show how a similar approach can be used in the determination of a lower bound for a more realistic model, namely the geometrically linearized cubic-to-tetragonal phase transition, in which a second order lamination is forced by the presence of affine boundary conditions. This is a joint work with Angkana Rüland.

Philipp Tscherner (University of Freiburg)

Modeling and Simulation of Paper Folding

Abstract: In this project foldable objects are studied by means of modeling, numerical simulation and geometrical analysis. For the latter, a relation between the fold angle, the geodesic and normal curvature of folding arcs is derived, giving rise to interesting geometrical observations of structures involving curved folds. To model objects including kinks, a 2D energy is identified as the Γ-limit of a 3D elastic energy that accounts for disontinuities of the deformation gradient along a prescribed curve. The reduced model is discretized by a discontinuous Galerkin method that allows for a practical description of foldable structures by neglecting gradient jumps of the deformation along the folding curve. An error estimate is derived for the linear model that describes configurations with small deflections and does not include the isometry condition. The corresponding nonlinear model is discretized using a reconstructed Hessian and a relaxation of the isometry condition. The resulting discrete elastic energy Γ-convergences to the continuous 2D elastic energy under appropriate density assumptions on smooth folded isometries. Various numerical experiments are carried out to study the physical behaviour of foldable objects like flytraps.

David Wiedemann (Universität Augsburg)

Homogenisation of processes in porous media with evolving microstructure

Abstract: Many processes in porous media can cause a change of the microstructure, which can affect strongly the effective material properties as for instance the permeability. In order to derive mathematically effective models, we transform the problem from the evolving domain into a substitute problem on a periodic substitute domain. There, we pass to the homogenisation limit and transforming the limit equations back yields an effective model for the process which is under consideration. In particular, we consider an advection–reaction–diffusion processes where the microstructure depends on the unknown concentration. Thus, the substitute problem on the periodic substitute domain becomes highly non-linear. Nevertheless, we can homogenise this problem rigorously and obtain after a back-transformation an effective model. It includes an advection–reaction–diffusion equation coupled with a Darcy law for evolving microstructure. Furthermore, the concentration is coupled with the domain evolution via an internal variable and a reference cell in the homogenisation limit. The effective diffusivity and permeability become time- and space-dependent and can be computed using the solutions of the corresponding cell problems.

Stephen Watson (University of Glasgow)

The L^2-gradient flow of the 3-well Aviles-Giga: emergent ordering dynamics, scaling symmetries & frustration

Abstract: tbd

Piotr Wozniak (Universität Münster)

Approximation schemes for energies of presribed fracture geometry

Abstract: One perspective in the study of metamaterials concerns the question of designing high-contrast composites with desired effective properties. More precisely one may contemplate possible microstructures of simplistic patterns which in the macroscopic scale would give rise to ductile materials favouring fracture only in accord with prescribed directions. In terms of variational modelling this objective is articulated as the asymptotic analysis, or Γ-convergence, of surface-degenerate Mumford-Shah energies. In this talk we propose and discuss the model functional set in the realm of SBV-type spaces along with the determining microgeometry. Within such formulations a general displacement field may admit highly irregular cracks in the reference domain. Therefore In order to rigorously derive the homogenised limit it is essential to come up with density techniques that approximate the surface energy by more regular competitors. We are going to point out some key challenges in proving the desired Γ-convergence and report on partial progress on the matter.

Rumin Zhang (Karlsruher Institut für Technologie)

Wave propagation in periodic waveguides: spectral decomposition and numerical simulation

Abstract: In this talk, we discuss acoustic waves propagation in periodic waveguides, from theoretical and numerical point of view. The spectral decomposition of solutions is presented, which shows exactly the physical structure of the waves. Based on this structure, an efficient numerical method is also introduced to simulate the wave phenomenon in periodic waveguides.
