HCM Workshop

"Nonlinear PDEs: Recent Trends in the Analysis of Continuum Mechanics"

July 17 – 21, 2023

organized by
Sebastian Hensel, Tim Laux, Kerrek Stinson

Abstracts: Invited Talks

Barbora Benešová (Charles University Prague)

Non-interpenetration in thin-film models

— joint work with D. Campbell, M. Kružík and S. Hencl (all Prague)

Abstract: Non-interpenetration is a well-known challenge for solid elastic materials combining analytical and geometrical aspects. In the bulk model, at least on the conceptual level, non-interpenetration is quite understood even if many challenges still remain open. In the thin film, the situation seems to be even less clear. Focusing on rods in the plane, I will introduce a possible concept of non-interpenetration and show a density and Γ-limit result.

Sergio Conti (University of Bonn)

On functions with bounded Hessian-Schatten variation

— joint work with Luigi Ambrosio and Camillo Brena

Abstract: The space of functions with bounded 1-Hessian-Schatten-norm has been proposed to study problems in the theory of inverse problems and machine learning. We prove an optimal density result, relative to the 1-Hessian–Schatten total variation, of continuous piecewise affine (CPA) functions using a construction based on a mesh whose local orientation is adapted to the function to be approximated. We show that not all extremal functions with respect to the p-Hessian–Schatten total variation are CPA.

Selim Esedoḡlu (University of Michigan)

On median filters for motion by mean curvature

Abstract: The median filter scheme is an elegant, monotone discretization of the level set formulation of motion by mean curvature. It turns out to evolve every level set of the initial condition by another

class of methods known as threshold dynamics. Based on this connection, we revisit median filters in light of recent work on the threshold dynamics method.

Julian Fischer (Institute of Science and Technology Austria)

Quantitative stochastic homogenization of energies in fracture mechanics

— joint work with Antonio Agresti and Nicolas Clozeau

Abstract: The quantitative homogenization of nonlinear random materials is by now rather well-understood in the case of uniformly convex energies. On the other hand, in the case of nonconvex problems - such as Griffith-type models in fracture mechanics - the theory is still in its infancy. In Griffith-type models for fracture, the formula for the homogenized fracture energy leads to a problem of finding surfaces of minimal energy in a random environment. We establish an algebraic rate of convergence (with respect to the length of correlations in the medium) for the energy of such minimizing surfaces in the case of statistically isotropic media. We then discuss possible implications concerning a quantitative homogenization theory for Mumford-Shah-like functionals with random coefficients.

Harald Garcke (University of Regensburg)

On the interaction of mean curvature flow and diffusion on evolving hypersurfaces

— joint works with (analytical part) Helmut Abels and Felicitas Burger (both University of Regensburg) and (numerical part) Charlie Elliott (University of Warwick) and Balàzs Kovàcs (University of Regensburg)

Abstract: We consider a geometric problem consisting of an evolution equation for a closed hypersurface coupled to a parabolic equation on this evolving surface. More precisely, the evolution of the hypersurface is determined by a scaled mean curvature flow that depends on a quantity defined on the surface via a diffusion equation. This system arises as a gradient flow of a simple energy functional. Assuming suitable parabolicity conditions, we derive short-time existence for the system. The proof is based on linearization and a contraction argument. For this, we parameterize the hypersurface via a height function and thus the system, originally defined on an evolving surface, can be transformed onto a fixed reference surface. The result is formulated in a classical sense, holds for the case of embedded and immersed hypersurfaces alike and provides an existence time independent of small changes in the initial surface. Afterwards, several properties of the solution are analyzed. Emphasis is placed on to what extent the surface in our setting evolves the same as for the usual mean curvature flow. To this end, we show that the surface area is strictly decreasing but give an example of a surface that exists for infinite times nevertheless. Moreover, mean convexity is conserved whereas convexity is not. Finally, we construct an embedded hypersurface that develops a self-intersection in the course of time.

Finally, we discuss how solutions can be computed numerically with the help of an evolving surface finite element discretization. We will discuss optimal error bounds and present numerical experiments illustrating the above discussed qualitative properties of the flow as well as the convergence behaviour. The analytical part is a joint work with Helmut Abels and Felicitas Burger (both University of Regensburg) and the numerical part is joint work with Charlie Elliott (University of Warwick) and Balàzs Kovàcs (University of Regensburg).

Flaviana Iurlano (Sorbonne Université)

Convergence of critical points for a phase-field approximation of cohesive fracture energies

— joint work with Marco Bonacini

Abstract: Variational models for cohesive fracture are based on the idea that the fracture energy is released gradually as the crack opening grows. Recently, [1] proposed a variational approximation via Gamma-convergence of a class of cohesive fracture energies by phase-field energies of Ambrosio-Tortorelli type, which may be also used as regularization for numerical simulations. In this talk, based on a joint work with Marco Bonacini, I will address the question of the asymptotic behaviour of critical points of the phase-field energies in a simplified one-dimensional setting.

[1] S. Conti, M. Focardai, F. Iurlano, *Phase field approximation of cohesive fracture models*, Ann. Inst. H. Poincar'e Anal. Non Linéaire **33** (2016), 1033–1067.

Stefan Neukamm (TU Dresden)

Derivation of a homogenized bending model for prestrained plates

— joint work with Klaus Böhnlein, Oliver Sander and David Padilla-Garza

Abstract: The presence of prestrain has a profound effect on the mechanical behavior of slender structures and can lead to complex equilibrium shapes that exhibit spontaneous bending and symmetry breaking. In the talk, we consider nonlinear bending plates with a prescribed, microheterogeneous prestrain. Our goal is to analyze the relationship between the microheterogeneity and the shape of the energy minimizers.

To this end, we present an approach that combines analytical and numerical methods. We start with a 3D nonlinear elasticity model for a periodic material occupying a plate-like domain of small thickness. We consider a spatially periodic pre-strain modeled by a multiplicative decomposition of the deformation gradient. By simultaneous dimension reduction and homogenization, we rigorously derive (in the Γ-limit of vanishing thickness and period) a homogenized nonlinear bending model for plates with an emergent spontaneous curvature term that can lead to non-flat equilibrium shapes. The effective properties of the plate model (bending stiffness and spontaneous curvature) are characterized by corrector problems. These allow to relate the microstructural configuration of the 3D plate to the shape of the minimizers of the effective 2D model. We study this microstructure-shape relationship analytically at the level of a parametrized sandwich panel with periodically distributed, prestressed fibers, and numerically for general composites. Our study reveals a rather complex dependence of the equilibrium shape on the considered parameters.

Riccarda Rossi (University of Brescia)

Balanced viscosity solutions to a multi-rate system for damage and plasticity

— joint work with Vito Crismale (Roma) and Giuliano Lazzaroni (Firenze)

Abstract: This talk revolves around a rate-independent system describing the interaction of damage and plasticity in a visco-elastic body originally proposed by Alessi, Marigo and Vidoli. Now, a quite common way to mathematically model rate-independent processes consists in viscously regularizing the associated evolutionary equations and analyzing the notion of rate-independent evolution obtained in the vanishing-viscosity limit. We address this task in the framework of the model for damage and plasticity under consideration, allowing for different scalings of the viscosity parameters for the

displacement, the damage parameter and the plastic strain. Hence, we discuss the types of rate-independent evolution arising from the vanishing-viscosity approximation.

Tomáš Roubíček (Czech Academy of Sciences)

Eulerian mechanics of finitely-strained viscoelastic continua

Abstract: In contrast to a traditional belief that the Lagrangian description (in a reference frame) is more suitable for solids while the Eulerian one (in actual deforming frames) suits better for fluids, I will use the Euler rate formulation for continuum mechanics and thermomechanics of solids. It brings both advantages (easy interaction with spatial fields like gravitational or electromagnetic and unified desription of viscoelestic solids and fluid) and disadvantages (open or deforming boundary problematic and nonlinear stress contributions and convective time derivatives involved). There is another belief (or rather experience) that, analytically, nonlinear continuum mechanics needs some higher-gradient theories (called non-simple material concepts) for rigorous analysis of resulting PDEs. I will discus how various options influence velocity dispersion and quality factor of propagation of waves in 1D linear models, and then use the concept of so-called multipolar 2nd-grade materials (i.e. higher-gradient in dissipative potentials) for the models in Eulerian description. The essential analytical role of this nonsimple viscosity is to ensure Lipschitz regularity of velocity field, which then avoids devlopment of spurious singularities in transports of involved quantities as mass density or deformation gradient or plastic distorsion etc.

[1] A. Mielke, T. Roubíček, U. Stefanelli: *A model of gravitational differentiation of compressible self-gravitating planets.* **Preprint**, arXiv no.2305.06232 and WIAS Berlin no.3015.

[2] T. Roubíček: *Visco-elastodynamics at large strains Eulerian.* Zeitschrift f. angew. Math. Phys. **73** (2022), art.no.80.

[3] T. Roubíček: *Quasistatic hypoplasticity at large strains Eulerian.* J. Nonlin. Sci. **32** (2022), art.no.45.

[4] T. Roubíček: *Thermodynamics of viscoelastic solids, its Eulerian formulation, and existence of weak solutions.* **Preprint**, arXiv no.2203.06080.

[5] T. Roubíček: *Interaction of finitely-strained viscoelastic multipolar solids and fluids by an Eulerian approach.* **Preprint**, arXiv no.2205.10814.

[6] T. Roubíček: *Landau theory for ferro-paramagnetic phase transition in finitely-strained viscoelastic magnets.* **Preprint**, arXiv no.2302.02850.

[7] T. Roubíček, U. Stefanelli: *Viscoelastodynamics of swelling porous solids at large strains by an Eulerian approach.* SIAM J. Math. Anal., in print, doi:10.1137/22M1474229.

[8] T. Roubíček, G. Tomassetti: *Inhomogeneous finitely-strained thermoplasticity with hardening by an Eulerian approach.* **Preprint**, arXiv no.2304.05918.

Richard Schubert (University of Bonn)

Convergence to the flat geometry for the Mullins-Sekerka evolution

— joint work with Felix Otto (Leipzig) and Maria G. Westdickenberg (Aachen)

Abstract: We consider the evolution of fairly wild perturbations of the plane in three space dimensions by the Mullins-Sekerka law. Mullins Sekerka is a non-local third-order geometric evolution that

is characterized by preservation of mass and reduction of surface area. Only assuming initial finiteness of the excess mass and the excess surface energy, we prove that the surface eventually becomes a graph, and that the energy converges with an optimal algebraic rate towards the flat ground state. I will discuss how the gradient flow structure and the L^1-method for conservation laws crucially enter the proof. Based on joint work with .

Sebastian Schwarzacher (University of Uppsala)

Regularity for fluid-structure interactions and its relation to uniqueness

— joint works with D. Breit, P. R. Mensah, B. Muha, M. Sroczinski and P. Su

Abstract: In the lecture I will survey recent progress on regularity estimates for solutions of Navier-Stokes equations interacting with an elastic shell. The shell is assumed to be perfectly elastic, which means that it is governed by a hyperbolic evolution. The deformation of the shell prescribes a part of the fluid domain, which makes the problem inherently non-linear. It will be shown how the "parabolic effect" of the fluid suffices to show results for weak solutions to the coupled fluid-structure interactions previously known for the Navier-Stokes equations in fixed domains. In particular I will discuss the validity of the so-called Ladyzhenskaya-Prodi-Serrin condition for regularity and weak-strong uniqueness for 3D Navier Stokes in the context of fluid-structure interactions.

Ulisse Stefanelli (University of Vienna)

The Poynting-Thomson model at finite strains

— joint work with Andrea Chiesa (Vienna) and Martin Kružík (Prague)

Abstract: I will discuss a variational model for the viscoelastic evolution at finite strains. In its linearized small-deformation limit, the model corresponds to the serial connection of an elastic spring and a Kelvin-Voigt viscoelastic element. In the finite-strain case, the total deformation of the body results from the composition of two maps, describing the deformation of the viscoelastic element and the elastic one, respectively. I will introduce a suitably weak notion of solution and prove existence by time-discretization. In addition, I will present a rigorous linearization result, showing that the corresponding small-strain model is indeed recovered in the small-loading limit.

Marita Thomas (FU Berlin)

Approximating dynamic phase-field fracture with a first-order formulation for velocity and stress

— joint work with Sven Tornquist (Berlin) and Christian Wieners (Karlsruhe), also based upon collaboration with Kerstin Weinberg and Kai Friebertshäuser (both Siegen) within the priority programme "Variational Methods for Predicting Complex Phenomena in Engineering Structures and Materials" (SPP 2256), project "Nonlinear Fracture Dynamics: Modeling, Analysis, Approximation, and Applications", financially supported by the German Research Foundation (DFG)

Abstract: We investigate a model for dynamic fracture at small strains. The sharp crack interface is regularized with a phase-field approximation. For the phase-field variable a viscous evolution with a quadratic dissipation potential is employed and a non-smooth penalization prevents material healing. For the solid material both the case of a visco-elastic and of a purely elastic constitutive law is considered. The momentum balance is formulated as a first order system and coupled in a nonlinear way to the non-smooth evolution equation of the phase-field variable. We introduce a full discretization in time and space, using a discontinuous Galerkin method for the first order system. Based on this, we show the existence of discrete solutions. We discuss their convergence to a suitable notion of weak solution of the system as the step size in space and time tends to zero and give a comparison to other formulations existing in literature. Simulation results are presented.

Ian Tobasco (University of Illinois Chicago)

Towards a generalized elastic model for kirigami metamaterials

— ongoing work with Paul Plucinsky (USC) and Paolo Celli (Stony Brook)

Abstract: Mechanical metamaterials are many body systems with elastic interactions and unusual bulk properties. Examples include origami with many panels, or kirigami systems which are the focus of this talk. A general kirigami system is made by cutting a lattice of holes from an elastic sheet. The result gains additional degrees of freedom called "mechanisms", which enrich the overall kinematics of the sheet. We propose a generalized elastic model for predicting a given kirigami system's bulk motion in response to loads, based on a careful study of the stress of interactions between neighboring panels. We prove a series of compactness theorems and minimum energy scaling laws to substantiate the model. Numerical simulations are also performed, matching the result of experiments.

Raghavendra Venkatraman (NYU, Courant Institute)

Interaction energies in liquid crystal colloids

Abstract: In this talk we report on recent rigorous results for colloidal suspensions in a liquid crystal background. In the deep nematic regime we discuss recent results obtained with Alama, Bronsard and Lamy on justifying the electrostatic analogy widely used in physics.
In the opposite "paranematic" regime, we report recent work with Golovaty, Taylor and Zarnescu on interaction energies between particles.

Abstracts: Contributed Talks

Matteo Fornoni (University of Pavia)

Optimal distributed control for a non-local tumour growth model

— joint work with Cecilia Cavaterra (University of Milan) and Elisabetta Rocca (University of Pavia)

Abstract: We consider a non-local variant of a diffuse interface model for tumour growth proposed by Hawkins-Daarud, van der Zee and Oden (2012). The evolving tumour is described through a phase-field approximation, taking values between -1 and 1, with extrema corresponding to the healthy and tumour phases. This model consists of a non-local Cahn-Hilliard equation for the tumour phase parameter, coupled to a reaction-diffusion equation for a nutrient (e.g. oxygen or glucose). Starting from well-posedness results proved by Frigeri, Lam and Rocca (2017), we first establish strong regularity properties for the solution of the system, with particular emphasis on how the terms related to chemotaxis and a possible viscous regularisation intervene. Then, we introduce an optimal distributed control problem, where the objective is to find an optimal chemotherapy that could lead the tumour to a desired configuration. Regarding this issue, we prove the existence of an optimal control and derive first-order necessary optimality conditions, by studying the Fréchet-differentiability of the control-to-state operator and the adjoint system. Finally, we discuss possible extensions and improvements.

Georg Heinze (TU Chemnitz)

Graph-to-local limit for the nonlocal interaction equation

Abstract: In this talk I will discuss a class of nonlocal partial differential equations defined in a tensor-weighted space that arise asymptotically from upwind-induced nonlocal dynamics on localising infinite graphs. The convergence of solutions of the graph equations to a solution of the corresponding local equation is achieved with variational methods, exploiting the equations gradient flow nature. The graph gradient structures are nonsymmetric, thus leading to Finsler-type graph gradient flows. In contrast, the limiting local gradient flow corresponds to a symmetric Otto-Wasserstein gradient structure, so that the presented graph-to-local limit is in fact also a nonsymmetric-to-symmetric limit.

Malte Kampschulte (Charles University Prague)

Inertial evolution in the face of (self-)collision

— joint works with A. Češík (Charles University Prague) and G. Gravina (Temple University)

Abstract: The aim of this talk is to show how recently developed methods allow us to apply variational approaches to fully dynamic (i.e. involving inertia), non-linear problems and to show existence of weak solutions purely from an energetical-approach. This permits us to deal with complicated, topologically nontrivial state-spaces and the resulting Lagrange-multipliers, which directly translate into physical forces. This will be illustrated at the example of viscoelastic solids, where we derive existence of weak solutions with correct collision-behaviour directly from the assumption of non-interpenetration of matter, without having to resort to soft potentials or other penalisation terms.

Frédéric Marazzato (Louisiana State University)

Computation of homogenized origami surfaces

Abstract: Origami folds have found a large range of applications in engineering, as solar panels for satellites, or to manufacture metamaterials. The talk focuses on solving a PDE developed in [Nassar et al, 2017], for the Miura fold, which is a classical origami pattern. We first study the constrained nonlinear elliptic PDE describing Miura surfaces, and prove the existence and uniqueness of solutions for ad hoc boundary conditions. Then, a finite element method, based on a least-square formulation and a Newton method, is proposed, to approximate the solutions. Finally, some numerical examples are presented.

Alice Marveggio (Institute of Science and Technology Austria)

Uniqueness and Stability of Planar Multiphase Mean Curvature Flow Beyond a Circular Topology Change

— work in progress with Julian Fischer, Sebastian Hensel and Maximilian Moser

Abstract: The evolution of a network of interfaces by mean curvature flow features the occurrence of topology changes and geometric singularities. As a consequence, classical solution concepts for mean curvature flow are in general limited to short-time existence theorems, which include singular times only for some stable shrinkers such as the circle. At the same time, the transition from strong to weak solution concepts (e.g. Brakke solutions) may lead to non-uniqueness of solutions.

Following the relative energy approach à la Fischer–Hensel–Laux–Simon, we prove a quantitative stability estimate holding up to the singular time. This implies a weak-strong uniqueness principle for weak BV solutions to planar multiphase mean curvature flow beyond circular topology changes.

Furthermore, we expect our method to have further applications to other types of shrinkers, as well as to prove quantitative convergence of diffuse-interface (Allen-Cahn) approximations for mean curvature flow.

Gokul Nair (Cornell University)

Energy Minimizing Configurations for Single-Director Cosserat Shells

Abstract: We consider a class of single-director Cosserat shell models accounting for both curvature and finite mid-plane strains. We propose a convexity condition for the stored-energy function that reduces to a physically correct membrane model in the absence of bending. Among other things, we argue that this model is suitable for predicting wrinkling. With appropriate growth conditions, we establish the existence of energy minimizers. The local orientation of a minimizing configuration is maintained via the blowup of the stored energy as a version of the local volume ratio approaches zero.

Willem van Oosterhout (WIAS Berlin)

Poro-visco-elastic solids at finite strains with degenerate mobilities

— joint work with Matthias Liero (WIAS Berlin)

Abstract: Poro-visco-elastic solids are materials in which a species diffuses and contributes to their deformation. Taking the Biot model as starting point, we discuss the modelling of (large) deformations

and diffusion processes in a physically consistent way, and show that under certain assumptions weak solutions exist.

To model poro-visco-elastic materials, a free energy functional and dissipation potential are introduced. This energy functional has contributions both from the deformation gradient, i.e., an elastic contribution, and from a mixing term, which couples the deformation gradient and the concentration of the species. The dissipation potential features contributions for the viscous evolution and the diffusion process. In particular, since the focus is on finite-strain elasticity, the diffusion equation has to be pulled back to the reference configuration. As the pulled-back mobility tensor depends nonlinearly on the deformation gradient, and the viscous stresses are frame-indifferent, it is analytically necessary to include the hyperstress as a higher-order regularization, which makes this a second order non-simple material.

Another important property of the investigated model is that it allows for degenerate mobilities, which are more physically relevant; however, this greatly complicates the analysis, and requires an additional regularization step. This is in contrast to prior work, where only non-degenerate mobilities were considered. Still, under certain assumptions on the constitutive laws and the data, the existence of weak solutions to the coupled system is shown using variational methods and fixed-point arguments.

David Padilla-Garza (TU Dresden)

Linking Energy to Irregularity in Prestrained Elastic Sheets

— joint work with Ian Tobasco

Abstract: This talk will delve into the link between prestrain regularity and energy in non-Euclidean thin elastic sheets. We show that in 2d stretching+bending models, prestrain irregularity implies a lower bound on the energy of the system. We also analyze the example of incompatible inclusions, a problem with applications in thin nematic elastomer sheets. In this example, we show a lower bound for general 3d elastic energy functionals and a matching upper bound, constructed by a combination of origami maps and Brehm isometries.

Arghya Rakshit (University of California, Irvine)

Sobolev regularity for optimal transport maps of non-convex planar domains

— joint work with Connor Mooney (arXiv:2306.08139)

Abstract: The regularity of optimal transport maps is a delicate matter that for the most part has focused on the case that the source and target domains are convex. In this talk we discuss a sharp global $W^{2,p}$ estimate for potentials of optimal transport maps that take a certain class of non-convex planar domains to convex ones.

Andrew Warren (University of British Columbia)

Gradient flow structure for some nonlocal diffusion equations

Abstract: We consider a family of nonlocal diffusion equations with a prescribed equilibrium state, which arise as the gradient flow of the relative entropy with respect to a version of the nonlocal Wasserstein metric introduced by Erbar. Such equations may also be viewed as the evolutionary Gamma-limit of a certain sequence of heat flows on discrete Markov chains. I will discuss criteria for

existence, uniqueness, and stability of solutions, and sufficient criteria on the equilibrium state which ensure fast convergence to equilibrium.