DEGENERATIONS OF HYPERKÄHLER VARIETIES

FALL TERM 2023, FRIDAY 10:30-11:30

1. Program

The format for the seminar follows the tradition of the working and discussion seminar in Bonn. Suggestions for other talks are more than welcome, maybe after consultation with some other participants to see whether the topic would be of interest. Ideally every talk should lead to a discussion afterwards. You should be willing to share your insight and maybe embark on a joint project. Possibly every talk should contain a set of problems and conjectures. Maybe those could be collected in an official problem list.

1.1. **Limiting mixed Hodge structure.**

(13 Oct, Philip Engel) We should review the classical work of Schmid [20]. One should state the Nilpotent orbit theorem 4.9 and SL_2-orbit theorem 5.13. After that, Theorem 6.16 should be explained. This theorem defines the limit mixed Hodge structure of a degeneration. It could be useful to give also some topological insight on the weight filtration, namely its relation with the monodromy and/or the Clemens retraction. If time permits (which is unlikely), we can discuss the idea of the proofs. The generalization of this work is contained in [2]. The notes [18], [21] and references therein may also be useful.

1.2. **Mixed Hodge structure of log symplectic pairs.**

(20 Oct, Mirko Mauri) Log symplectic pairs appear as irreducible components of the degenerations of hyperkähler varieties. Explain Theorem 1.2 and Theorem 1.7 in [8]. Check if the assumption on good degeneration can be removed, and if the simple normal crossing assumption can be replaced with divisorial log terminal.

1.3. **Degenerations of type I.**

(27 Oct, Dominique Mattei) If an irreducible component of the central fiber of a degeneration is not uniruled, then the monodromy is finite, and up to an alteration the degeneration is in fact a smooth deformation. Explain the ideas behind Theorem 0.2, 0.3, 0.6, 0.7 in [13]. The previous result is false for arbitrary Calabi–Yau varieties (what about abelian varieties?). In this regard, do not forget to comment on Remark 0.8 in [13]. If time permits (which is unlikely) discuss the applications to special examples of hyperkähler manifolds contained in Section 5.

1.4. **Dual complexes of degenerations of hyperähler manifolds.** (3 Nov, Anna Abasheva) Define the notion of dual complex (or essential skeleton); see for instance [4, §2]. Explain the relation between the dimension of the dual complex and the index of the unipotent monodromy, which determine the type of the degeneration; see [17, Theorem 4.2.4. (3) and (4)]. Describe fundamental group [14, §3.4] and cohomology of the dual complex ([13, Thm 6.14 and Prop. 6.15]; check why the cup product on the singular cohomology of the dual complex matches the ring structure of the limiting mixed Hodge structure). The main reference together with the citations therein is [13, §6].[1] The dual complex of a type III degeneration of hyperkähler of dimension $2n$ is conjectured to be homeomorphic to $\mathbb{P}^n$: a conjectural explanation appears in [13, §6.2] and [10, §3.5]; mention the SYZ conjecture. The dual complex of a type III degeneration is conjectured to be a standard simplex of dimension n: I am not aware of any conjectural explanation besides explicit examples and cohomological computations. Examples of dual complexes of degenerations of hyperkähler and partial evidence for the previous conjectures are provided in [1, Thm 1.7.1 and 1.7.2]. See also [15, §5.2 and §5.3] and [7].

1.5. **P=W for compact hyperkähler manifolds.** (17 Nov, Andres Fernandez Herrero) The talk is an exposition on [9]; see also [10, Section 3]. If time permits, one may show that given any irreducible holomorphic symplectic variety X with $b_2 \geq 5$, there exists a projective type III degeneration of irreducible holomorphic symplectic varieties, deformation equivalent to X; see [24, Theorem 4.6]. Adapting the proof of [24], one could prove the following: let X be an irreducible holomorphic symplectic variety, and assume that $(H^2(X, \mathbb{Z}), q)$ contains an isotropic plane. Then there exists a projective type II degeneration of irreducible holomorphic symplectic varieties, deformation equivalent to X.

1.6. **Degenerations of type III.** (1 Dic, Mauro Varesco) The limiting mixed Hodge structure of a maximal degeneration of hyperkähler manifolds is of Hodge–Tate type. This relies on the behaviour of the Kuga–Satake construction under degeneration. Focus on this construction; see Theorem 1.1 in [22] and Proposition 3.7 in [24]. As a corollary, we obtain Proposition 3.8 in [24].

1.7. **Smoothing of simple normal crossing divisors.** (8 Dic, Roberto Svaldi) The special fiber of a degeneration of hyperkähler manifolds is a degenerate hyperkähler variety. Viceversa, one may dream to smooth well-chosen simple normal crossing varieties to possibly new deformation type of hyperkähler manifolds. Not all simple normal crossing divisors admit a smoothing, but d-semistable do. The goal of this talk is to explain the definition of d-semistability for a simple normal crossing with trivial canonical bundle, and check that indeed it can be smoothed. Present the proof of Namikawa and Kawamata; see [12, Theorem 4.2]. Other references are [5, 3, 19] and references therein.

[1] A major open problem is to provide a computable and algebraic interpretation of the torsion of the integral homology of the dual complex of any irreducible component of the central fiber.

1.8. **Degeneration of type II: Nagai conjecture.** (15 Dic, Giacomo Mezzedimi) State the Nagai conjecture and show that it holds for degeneration of type I and III. Follow the references in the introduction of [11]. Partial results are contained in [16], [6], [8], [11]. Two alternative characterizations of the Nagai conjecture are provided in [6, Proposition 1.16] and [11, Corollary 3.4] (the latter is equivalent to the existence of a Hodge octahedron - talk with Mirko!; see also [23, Remark 4.3]).

2. List of problems or conjectures

- (M. Mauri) Fix an even integer n. Show that there exists a constant $C = C(n)$, only depending on n, such that the order of an automorphism of an irreducible holomorphic symplectic variety of dimension n acting trivially on $H^2(X, \mathbb{Q})$ is bounded above by C.
- (C. Lehn) Assume $f\colon X \dashrightarrow B$ is a Lagrangian fibration of an irreducible holomorphic symplectic manifold. Show that B is factorial (even smooth). Is the singular locus of B contained in the divisorial part of the discriminant locus (this is false if X is singular)?
- (C. Lehn) Find examples of isotrivial Lagrangian fibrations on hyperkähler manifolds of OG6 and OG10 type.
- (M. Mauri) Let X be an irreducible holomorphic symplectic variety endowed with a Lagrangian fibration. Is it possible to find an irreducible holomorphic symplectic variety, deformation equivalent to X, endowed with an isotrivial Lagrangian fibration?

More questions and conjectures are welcome!

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