Families of algebraic varieties
Felix Klein Lectures

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Main question, old style

– Can we parametrize all **varieties** (or sheaves, morphisms, ...) in a **natural** way?

Main questions, new style

– What is a good family of algebraic varieties?
– Can we describe all good families in an optimal manner?

Plan of the lectures

- History and examples, from Riemann to Mumford
- Moduli of varieties; main questions and definitions
- Characterizations of stable families
- Du Bois property and consequences
- K-flatness
- Difficulties in positive characteristic

— The lectures will be mostly independent of each other.

— For details, see mainly the books

Singularities of the minimal model program, CUP, 2013

Families of varieties of general type, CUP, 2023

Families of algebraic varieties
Felix Klein Lecture # 1
János Kollár

History and examples,
from Riemann to Mumford

Example – Hypersurfaces

– $X_d \subset \mathbb{P}^n$ of degree d.
– Equation: $\sum_I a_I x^I = 0$ where
$I = (i_0, \ldots, i_n)$ and $i_0 + \cdots + i_n = d$.

Classical claim. All degree d hypersurfaces in $\mathbb{P}^n$ "naturally" form a projective space $\mathbb{P}^N$ where $N = \binom{n+d}{n} - 1$:

$$X_d = \left(\sum_I a_I x^I = 0\right) \leftrightarrow \{a_I\}.$$

– works over any field
– counts multiplicities
– similar: hypersurface sections of any $Y^n \subset \mathbb{P}^M$.

Hypersurfaces with coordinate changes

Claim. Let $X_i \subset \mathbb{P}^n$ be hypersurfaces and $\phi : X_1 \cong X_2$ an isomorphism. Then ϕ extends to a linear coordinate change $\Phi : \mathbb{P}^n \cong \mathbb{P}^n$, except possibly in the following cases

- $n = 1$
- $n = 2$ and $\deg X_i \leq 3$ (Castelnuovo, Serrano)
- $n = 3$ and $\deg X_i = 4$ (needs Lefschetz)

Aside: Determinantal examples

$W \subset \mathbb{P}^n_x \times \mathbb{P}^n_y$: intersection of $n+1$ bidegree $(1,1)$:
$\sum_{ijk} a^k_{ij} x_i y_j = 0$.

Projections:
$W_x = \left(\det(\sum_i a^k_{ij} x_i) = 0\right) \subset \mathbb{P}^n_x$ and
$W_y = \left(\det(\sum_j a^k_{ij} y_j) = 0\right) \subset \mathbb{P}^n_y$.

Oguiso (2017): For $n = 3$ we get smooth degree 4 surfaces, that are not even Cremona equivalent.

One should study:

$$\mathrm{Hyp}_{d,n} := \{\text{Hypersurfaces of degree } d \text{ in } \mathbb{P}^n\}/\mathrm{PGL}_{n+1}.$$

$\mathrm{Hyp}_{d,n}$ is a horrible space

Closure of a subset $U \subset \mathrm{Hyp}_{d,n}$:

given $X_t := \big(F(x_0, \dots, x_n; t) = 0\big)$,

if $[X_t] \in U$ for $t \neq 0$ then $[X_0] \in \bar{U}$.

Fix $X := F(x_0, \dots, x_n)$ and let

$F(x, t) := F(x_0, \dots, x_r, tx_{r+1}, \dots, tx_n)$.

- $X_t \cong X$ for $t \neq 0$ and
- $X_0 = F(x_0, \dots, x_r, 0, \dots, 0)$.

$r = 0$ case gives:

Corollary. $[(x_0^d = 0)]$ is the only closed point of $\mathrm{Hyp}_{d,n}$.

Trying to fix it

- $\mathrm{Hyp}_{d,n}^{\mathrm{reduced}}$
  only closed points are $[F(x_0, x_1, 0, \ldots, 0) = 0]$.
- $\mathrm{Hyp}_{d,n}^{\mathrm{normal}}$
  only closed points are $[F(x_0, x_1, x_2, 0, \ldots, 0) = 0]$.

(The above are all cones with large singular sets.)

- $\mathrm{Hyp}_{d,n}^{\mathrm{isolated,non-cone}}$ example:

$$X_t := (x_0^{d/2} + t^{d/2} x_1^{d/2}) x_1^{d/2} + x_2^d + \cdots x_n^d$$

– $X_t \cong X_1$ if $t \neq 0$ (apply $(x_0, x_1) \mapsto (tx_0, t^{-1}x_1)$)
  1 isolated singularity

– X_0: 2 isolated singularities of multiplicity $d/2$.

GIT of Hypersurfaces, Hilbert and Mumford

There is a notion of stability.

- $\mathrm{Hyp}_{d,n}^{\mathrm{stable}}$ is as nice as possible:
  noncompact, nearly smooth algebraic variety, and
- $\mathrm{Hyp}_{d,n}^{\mathrm{semistable}}$ is less nice but
  compact algebraic variety.

Good property: smooth $\Rightarrow$ stable.

Bad properties:

– not clear what else is stable if $d \geq 4$
– semi-stable points correspond to
many different hypersurfaces.

Smooth limits of hypersurfaces (Mori, 1975)

Consider $\deg G(x) = d, \deg F(x) = de$, $\deg z = d$ and

$$X_t := \left(z^e - F(x) = G(x) - tz = 0\right).$$

- for $t \neq 0$: X_t smooth hypersurface of degree de

$$X_t := \left(G^e(x) - t^e F(x) = 0\right).$$

- for $t = 0$: X_0 is not a hypersurface but a degree e cover of $(G = 0)$ ramified along $(F = 0)$.

Question. Any prime degree examples for $\dim \geq 3$?

Ottem–Schreieder (2020): no for degrees 5 and 7.

Plane curve version: DeVleming–Stapleton (2022)

Aside: $n+1$ points $p_j \in \mathbb{C}$ up to translations

Coordinates a_i using $x^{n+1} + a_2 x^{n-1} + \cdots + a_{n+1}$.

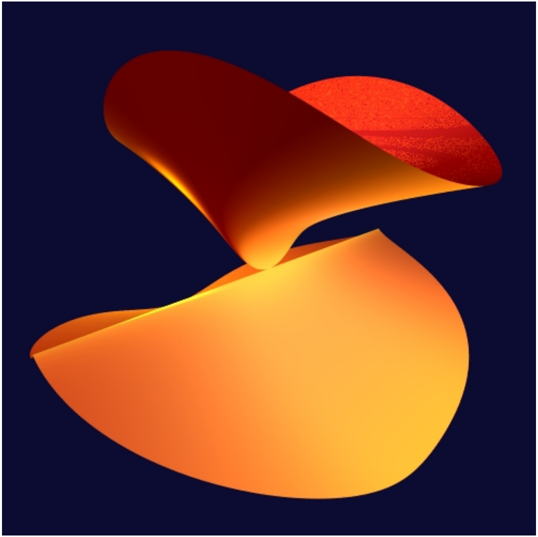
Look at $p_0, \ldots, p_n \in \mathbb{C}$ where
 P_n^u:= at least n of the points coincide, or
 P_n^m:= plus $q \in \mathbb{C}$ such that $p_i = q$ at least n-times.

If $n \geq 2$ then q is determined by $p_0, \ldots, p_n$, yet

Claim: $P_n^m \cong \mathbb{C}$ **but** P_n^u is a cuspidal curve.
Pf: $(p_0, \ldots, p_n; q) \mapsto \sum_j (p_j - q) \in \mathbb{C}$ gives $P_n^m \cong \mathbb{C}$.
If the n-fold point is at t, then $(x-t)^n(x+nt)$. So
$a_i = c_i t^i$, and P_n^u = image of $t \mapsto \left(c_2 t^2, \ldots, c_{n+1} t^{n+1}\right)$.

Higher dimensional version:
$xy + z^{n+1} + tz^n$ is trivial to first order.

Questions?



Moduli of curves, analytic theory I

Riemann (1857), *Theorie der Abel'schen Funktionen*
Riemann surfaces of genus g depend on $3g - 3$ parameters

Fricke–Klein (1897-1912), *Vorlesungen über die Theorie der automorphen Funktionen*, (1300 pp.)
T_g exists and is contractible:

Siegel (1935), construction of A_g as analytic space

very precise, modern feel, mostly arithmetic

Red Herring - falsche Spur - fausse piste - falsa pista

$T_g =$ discrete, cocompact representations
$\pi_1(C) \to \mathrm{PGL}_2(\mathbb{R}) = \mathrm{Aut}(\text{unit disc})$, modulo conjugation

This is a **real** manifold (of real dim 6g-6)

Complex structure not natural, not considered much in Fricke–Klein.

Aside. Weil knew that the
$H^{p,q} \subset H^{p+q}(X, \mathbb{C})$ vary **real** analytically with X.

Griffiths: the filtration varies **complex** analytically with X.

Moduli of curves, analytic theory II

Teichmüller (1940–44), complete theory of T_g

complex structure + functorial aspects.

Weil (1958), Bourbaki seminar: "As for M_g there is virtually no doubt that it can be provided with the structure of an algebraic variety"

Grothendieck (1960), Cartan Seminar (after Teichmüller?)
T_g represents a functor:
projective families over analytic bases

Worth reading: A'Campo–Ji–Papadopoulos (2016):
On the early history of moduli and Teichmüller spaces

Moduli of curves, algebraic theory I

Cayley (1860/62), *A new analytic representation of curves in space*[1]. Constructs moduli of space curves.

$C \mapsto$ (all lines meeting C)

General theory: van der Waerden, Chow, Hodge-Pedoe

Hilbert (1890), *Über die Theorie der algebraischen Formen.*

Finite generation of rings of invariants.

("Theologie" according to Gordan.)

BUT: nobody seems to have taken its Proj

Hurwitz (1891), *Über Riemann'sche Flächen mit gegebenen Verzweigungspunkten.* M_g is irreducible

[1] 2 papers, same title

Moduli of curves, algebraic theory II

Severi (1915), *Sulla classificazione delle curve algebriche e sul teorema d'esistenza di Riemann.*

M_g unirational for $g \leq 10$.

Existence? Not clear what he thinks, uses "Mannigfaltigkeiten" (after Riemann) not "varietà".

Claim: there is a family over a rational variety that gives almost all curves of a fixed genus.

Weil, Matsusaka (1946–56) field of definition/field of moduli

M_g, A_g should be defined over $\mathbb{Z}$, so

$k_C :=$ residue field of $[C] \in M_g$.

Aim: finding k_C from C (without knowing M_g).

Moduli of curves, algebraic theory II

Satake (1956-60): Compactifying A_g, by viewing it as quotient of a symmetric domain:

$$\overline{A}_g = A_g \amalg A_{g-1} \amalg A_{g-2} \amalg \cdots \amalg A_0.$$

Baily-Borel (1966) (general symmetric domain case).

Red Herring II

Over $\overline{A}_g \setminus A_g$ the natural objects are **lower dimensional** Abelian varieties.

There is no 'natural' flat family of g-dimensional varieties
— over $\overline{A}_g$. Not even
— over any $Z \stackrel{u}{\twoheadrightarrow} \overline{A}_g$, with u quasi-finite and dominant.

Alexeev (2002) gave the first compactification of A_g with a 'natural' modular interpretation.

Moduli and compactification using GIT

Mumford (1965): M_g

Mumford, Gieseker (1974-80) $\bar{M}_g$

Gieseker (1977): moduli of (canonical models of) surfaces, for high enough pluricanonical embedding,

Viehweg (1989–95): higher dimensional canonical models, with well chosen polarization.

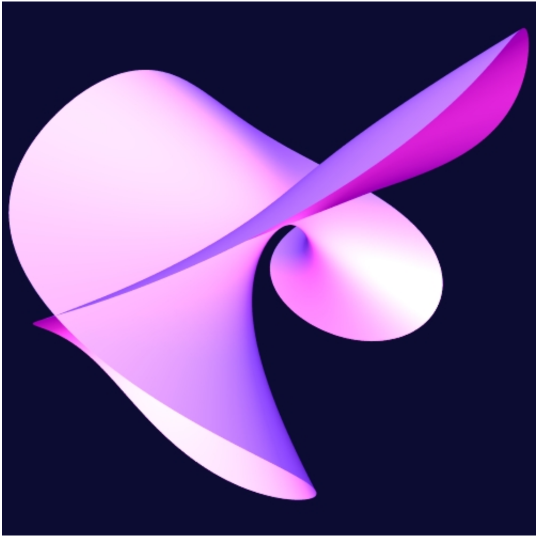
Red Herring III

Mumford and the others were too strong technically.
They made the 'linear' GIT work for canonical models, but GIT breaks down at the boundary.

First hint: (Mumford, 1977)
asymptotic stability of singularities:
seems pretty random class.

Definitive answer: Xiaowei Wang – Chenyang Xu (2012)
GIT compactification of the moduli of surfaces
forever depends on the pluricanonical embedding,
(both Chow and Hilbert versions).

Questions?



Genus 2 curves or $\mathrm{Hyp}_{6,1}$

- C: smooth, projective curve of genus 2, or
  smooth, compact Riemann surface of genus 2.

Structure theorem. There is a unique $\tau : C \to \mathbb{P}^1$ of degree 2 ramified at 6 points.

Equation: $z^2 = f_6(x, y)$ (no multiple roots)

Corollary. M_2, the set/space of all smooth, projective curves of genus 2 is

- $\{6 \text{ points in } \mathbb{P}^1\}/\mathrm{PGL}_2$, equivalently
- $\left(\mathrm{Sym}^6\mathbb{P}^1 \setminus \text{diagonals}\right)/\mathrm{PGL}_2$.

Compactifying M_2

Typical example: 4-fold root for $t = 0$ at $(0{:}1)$:

$$f_6(x{:}y, t) = (x - ta_1y)\cdots(x - ta_4y)(x - a_5y)(x - a_6y)$$

Coordinate change $x = tx', y = y'$ and dividing by t^4:

$$(x' - a_1y')\cdots(x' - a_4y')(tx' - a_5y')(tx' - a_6y')$$

which has ony 2-fold root at $(1{:}0)$

Lemma. Same trick achieves: at most triple root at $t = 0$.

Triple root case: (after base change)

$$(x - t^2a_1y)\cdots(x - t^2a_3y)(x - a_4y)\cdots(x - a_6y).$$

$x = tx', y = y'$ and dividing by $t^3a_4a_5a_6$ we get

$$(x' - ta_1y')\cdots(x' - ta_3y')(\tfrac{t}{a_4}x' - y')\cdots(\tfrac{t}{a_6}x' - y').$$

For $t = 0$ this becomes

$(x')^3(y')^3$: two triple roots.

GIT compactification $\bar{M}_2^{\mathrm{GIT}}$

Points correspond to:
•: two triple roots (unique point) and
•: at most double roots.

Corresponding curves:
•: $z^2 = x^3y^3$ rational with 2 cusps.
•: at most double roots $z^2 = f_6(x, y)$.
Irreducible with at most nodes, except:

• $z^2 = x^2(x-y)^2(x+y)^2$. Contract one of the components: rational with 1 triple point like the 3 coordinate axes.

End of old style story.

$\bar{M}_2^{\mathrm{GIT}}$ is a very unpleasant compactification.

- Local universal families:

  At 2 cusp point $z^2 = x^3y^3$, deformations are

  $z^2 = \left(x^3 + uxy^2 + vy^3\right)\left(y^3 + syx^2 + tx^3\right).$

Problem: $(u = v = 0)$ or $(s = t = 0)$ define disallowed curves.

- Stacky problem at $z^2 = x^2(x-y)^2(x+y)^2$.

Deligne–Mumford compactification $\bar{M}_2$

- at most double roots $z^2 = f_6(x, y)$: keep these.
- $z^2 = x^2(x - y)^2(x + y)^2$: keep as is.
- $z^2 = x^3y^3$ change to:

double cover of pair of intersecting lines,
ramified at 3+3 pts plus the node:
= two elliptic curves meeting at a point.

Source of triple root problem: 3 choices

- contract one elliptic curve, or
- contract other elliptic curve, or
- blow up intersection point and contract both.

Deligne–Mumford compactification $\bar{M}_g$

Stable curves:

Projective, connected, reduced curves C such that:

Local: at worst nodes: $(xy = 0)$ (locally analytically)

Global: ω_C is ample.

What is ω_C?

– smooth curve: $\omega_C = \Omega_C = T_C^* = \mathcal{O}_C(K_C)$.

– for any plane curve, Poincaré residue map

$$\Re : \omega_{\mathbb{P}^2}(C)|_C \cong \omega_C$$

– if $C = \cup_i C_i$ and $P_i \subset C_i$ are the nodes then

$$\omega_C|_{C_i} = \omega_{C_i}(P_i).$$

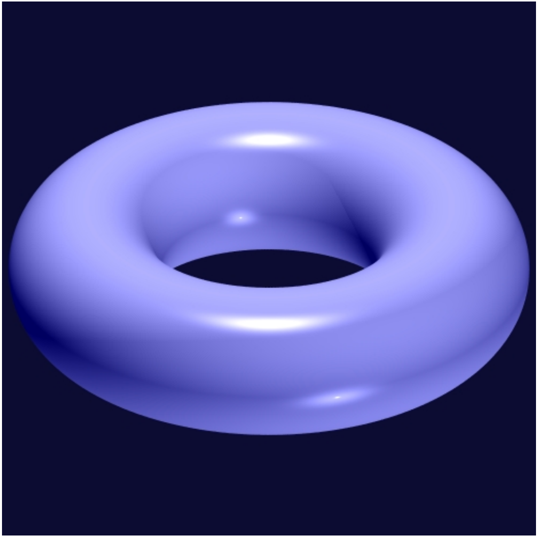
Higher dimension, basic questions

What are the correct analogs of
smooth, projective curves of genus ≥ 2?

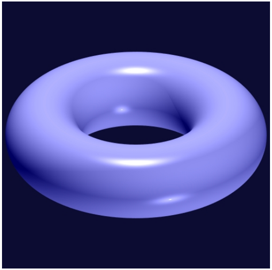
What are the correct analogs of stable curves?

Usually:
EASIER: make it work for an open moduli space.
HARDER: make it work for a compact moduli space.

Any questions?



Special homework



After plane and sphere,
this is the 3rd best known algebraic surface.

What is it?

Families of algebraic varieties
Felix Klein Lecture # 2
János Kollár

Moduli of varieties;
main questions and definitions

Higher dimension, basic questions

Question 1. What are the correct analogs of smooth, projective curves of genus ≥ 2?

Question 2. What are the correct analogs of stable curves?

Question 3. What are the correct analogs of flat families of stable curves?

Canonical models 1

X smooth, proper. Fix $m \geq 1$ and
any basis $s_0, \ldots, s_{N(m)} \in H^0(X, \omega_X^m)$.
Get a map $\phi_m : X \dashrightarrow X_m \subset \mathbb{P}^{N(m)}$.

Theorem (Iitaka, 1971) For m sufficiently divisible, the closed images X_m are

- birational to each other, and
- $X \dashrightarrow X_m$ has connected fibers.

Definitions

- *Kodaira dimension:* $\kappa(X) := \dim X_m$,
- *general type:* $X \dashrightarrow X_m$ birational.

Canonical models 2

A basis $s_0, \ldots, s_{N(m)} \in H^0(X, \omega_X^m)$ gives
a map $\phi_m : X \dashrightarrow X_m \subset \mathbb{P}^{N(m)}$.

Theorem (Canonical models)

For m sufficiently divisible, the closed images X_m are isomorphic to the canonical model of X:

$$X^{\mathrm{can}} := Proj \oplus_m H^0(X, \omega_X^m).$$

Note: True for any X, but 'canonical model' mostly used for general type only.

Canonical models, history

Finite generation of $\oplus_m H^0(X, \omega_X^m)$:

– $\dim X = 2$: Castelnuovo, Enriques ($+$ Mumford)
– $\dim X = 3$: Mori ($+$ Kawamata, Kollár, Reid, Shokurov) (1980–88)
– $\dim X \geq 4$: Shokurov, Corti, Hacon–McKernan (2003–09), Birkar–Cascini–Hacon–McKernan (2010), Fujino-Mori (2000).

Open question

Version 1. Are the $h^0(X, \omega_X^m)$ deformation invariant?

Version 2. Is there a natural transformation

$$\left\{ \begin{array}{c} \text{smooth families} \\ \text{of varieties of} \\ \text{general type} \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} \text{flat families of} \\ \text{canonical models} \end{array} \right\}?$$

Known cases in char 0:
Surfaces (classical + Iitaka, 1971)
Threefolds (Kollár–Mori, 1992)
Projective families over **reduced** base (Siu, 1998)
X_0 projective, **reduced** base (K., 2021)

Lecture 6 for char p.

ω on a singular variety I.

Recipe: (if X is normal)

Take smooth locus $X^\circ \subset X$

$\omega_{X^\circ} = \Omega^n_{X^\circ} = (\det T_{X^\circ})^*$, then extend it to X.

Powers: $\omega_X^{[m]} :=$ extension of $\omega^m_{X^\circ} = (\omega_X^{\otimes m})^{**}$.

Exercise: A line bundle L° on X° has at most 1 extension to a reflexive sheaf L on X, but it may have infinitely many extensions as a topological line bundle.

ω on a singular variety II.

- Hypersurfaces: $(g = 0) \subset \mathbb{A}^n$. Generator of ω:

$$(-1)^i \frac{dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n}{\partial g / \partial x_i}$$

- Quotients: $\mathbb{A}^n/(\text{finite group } G)$. Generator of $\omega^{[m]}$:

$$(dx_1 \wedge \cdots \wedge dx_n)^{\otimes m}$$

where $m = |G/G \cap \mathrm{SL}_n|$.

Canonical models: internal definition

Definition (Canonical singularity, Reid, 1980)

One can pull back pluricanonical forms. That is,
$p : Y \to X$ *resolution, then*

1. $K_Y \sim p^* K_X + (\mathit{effective})$, *equivalently, we have*
2. $p^* \omega_X^{[m]} \to \omega_Y^{[m]} \quad \forall m \geq 0.$

Definition (Canonical model)

Normal, projective with

1. *canonical singularities, and*
2. ω_X *is ample.*

Answer to Question 1

Thesis

Canonical models are the correct higher dimensional analogs of smooth, projective curves of genus ≥ 2.

Toward stable varieties 1

Lemma. B smooth curve, $B^\circ = B \setminus \{0\}$
$f^\circ : X^\circ \to B^\circ$ a family of canonical models.

There is at most 1 extension to

$$\begin{array}{ccc} X^\circ & \subset & X \\ f^\circ \downarrow & & \downarrow f \\ B^\circ & \subset & B \end{array}$$

such that

– ω_X (or $\omega_{X/B}$) is ample on every fiber, and
– X has canonical singularities.

Question. How to guarantee the latter?

Toward stable varieties 2

Needed in general case: $0 \in D = X_0 \subset X$, Cartier divisor.
Assume $X \setminus D$ has canonical sings and D has $\boxed{????}$
$\Rightarrow X$ has canonical sings.

Curve case: node $(xy = 0)$ is not canonical, but
$(xy + t^n = 0)$ is canonical $\forall n$.

Definition: $\boxed{????}$ = semi-log-canonical.

What is semi-log-canonical?

What is a node?

Generating section σ of ω_C for $C := (xy = 0) \subset \mathbb{C}^2$ is

$$\sigma = \frac{dx}{x} \text{ on } x\text{-axis}, \qquad \sigma = -\frac{dy}{y} \text{ on } y\text{-axis}.$$

Characterizations of nodes:

Using resolutions: $p : C' \to C$ then $p^*\sigma$ has only simple poles.

Using local volume: Although the local volume is

$$\frac{i}{2\pi} \int_{|x| \leq 1} \frac{dx}{x} \wedge \frac{d\bar{x}}{\bar{x}} = \infty,$$

it has only logarithmic growth:

$$\frac{i}{2\pi} \int_{|x| \leq 1} |x|^{\epsilon} \frac{dx}{x} \wedge \frac{d\bar{x}}{\bar{x}} < \infty \quad \text{for } \epsilon > 0.$$

Definition of semi-log-canonical = slc

- Deminormal:= X only nodes in codimension 1 and S_2
  (so ω_X is a line bundle in codim 1),
- $\omega_X^{[m]}$ is locally free for some $m > 0$ (with section σ^m),
- Three equivalent versions:

— Using resolution I: $K_Y \sim p^* K_X + (\text{effective}) - E$,
where $E =$ reduced exceptional divisor.

— Using resolution II: there is $p^*\omega_X^{[r]} \to \omega_Y^{[r]}(rE) \quad \forall r \geq 0$.

— Using local volume: $\int_X \sigma \wedge \bar{\sigma}$ has only

logarithmic growth:= $\left| \int_X |g|^\epsilon \cdot \sigma \wedge \bar{\sigma} \right| < \infty$,

for every g vanishing on $\operatorname{Sing} X$ and $\epsilon > 0$.

Proof of ???? = semi-log-canonical

$D \subset X$ Cartier divisor, $p : Y \to X$ resolution. Write $K_Y = f^* K_X + J$ and $f^* D = D_Y + E_D$. Note that

- $J \geq 0$ iff X has canonical singularities, and
- (after base change) all coefficients in E_D equal 1.

Adjunction formula: $K_{D_Y} =$

$$\big(K_Y + D_Y\big)|_{D_Y} = \big(f^*(K_X + D) + J - E_D\big)|_{D_Y} = f^* K_D + (J - E_D)|_{D_Y}$$

Suggests: $J \geq 0 \quad \Leftrightarrow \quad (J - E_D)|_{D_Y} \geq -1.$

Almost what we want, but no information on exceptional divisors that are disjoint from D_Y.

Convexity of the coefficients of J settles the rest. (Shokurov, Kollár, Kawakita, de Fernex-K-Xu)

Examples of semi-log-canonical singularities

- dim=2, normal: cone over elliptic curves (or cusps)
- dim=2: $(xy = 0)$, $(xyz = 0)$, $(y^2 = zx^2)$.
- dim=n examples:

  cone over $X \subset \mathbb{P}^N$ is lc (or slc) iff X is lc (or slc) and $-K_X \sim rH$ for some $r \geq 0$.

  cone over $X \subset \mathbb{P}^N$ is canonical iff X is canonical and $-K_X \sim rH$ for some $r \geq 1$.

Aside. Local π_1 of log-canonical sings?
old quess: almost solvable (even polycyclic)
no: surface groups (K. 2013)
maybe everything? Figueroa–Moraga (2022)

Answer to Question 2

Definition (Stable variety)

1. *Deminormal (at worst nodes in codim 1 and S_2),*
2. *semi-log-canonical singularities,*
3. *projective and ω_X is ample.*

Thesis

Stable varieties are the correct higher dimensional analogs of stable curves of genus ≥ 2.

Existence of stable limits 1

Kollár – Shepherd-Barron approach (1988):

B smooth curve, $p : Y \to B$ such that
- generic fiber smooth, general type, and
- all fibers reduced snc (=simple normal crossing).

(can be done using semistable reduction)

Let $p^{\mathrm{can}} : Y^{\mathrm{can}} \to B$ be the relative canonical model.

Claim. All fibers of $p^{\mathrm{can}} : Y^{\mathrm{can}} \to B$ are stable varieties.

Existence of stable limits 2

Problem: MMP does not work for
simple normal crossing varieties (K, 2011)

Solution:
normalize, take stable limits and then glue (K. 2013)

Quite difficult

Gluing example – triangular pillows

Glue 2 copies of $\mathbb{P}^2$ along axes $(xyz = 0)$:
$\sigma : (x{:}y{:}0) \mapsto (x{:}\lambda y{:}0)$,
$\sigma : (0{:}y{:}z) \mapsto (0{:}y{:}\mu z)$, and
$\sigma : (x{:}0{:}z) \mapsto (\nu x{:}0{:}z)$.

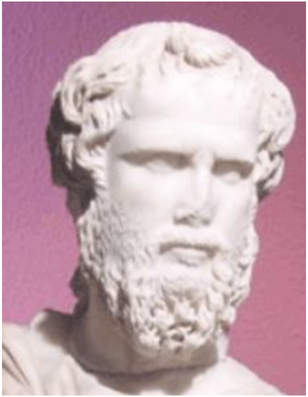
Question. When is $\mathbb{P}^2 \amalg_\sigma \mathbb{P}^2$ projective?

Answer. $\lambda\mu\nu$ is a root of unity.

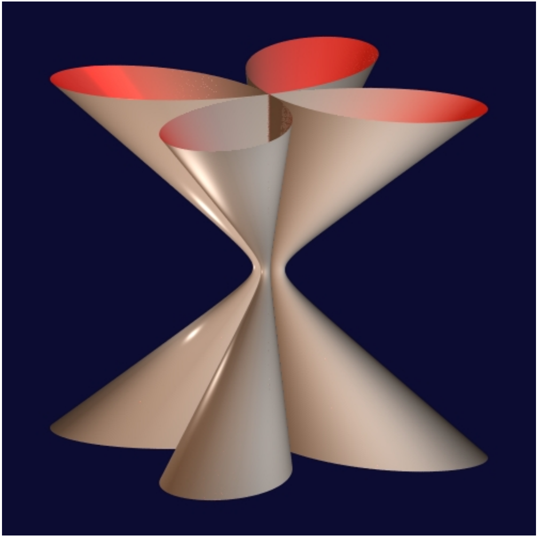
Answer. $\lambda\mu\nu$ is a root of unity.

The $\mathbb{P}^2 \amalg_\sigma \mathbb{P}^2$ are reducible K3 surfaces, usually non-projective.

Menelaus of Alexandria
$\sim$ 70-140 AD
(did degree 2 case)



Questions?



About Question 3

Question 3. What are the correct analogs of flat families of stable curves?

Thesis

Flat families of stable surfaces is the wrong answer.

Flatness not enough, example 1

Surfaces with quotient singularities and ample K

- appear at the boundary of the moduli of smooth surfaces, but
- can have very bad flat deformations.

Goes back to Bertini:

The cone over the degree 4 rational normal curve has 2 types of smoothings

- to Veronese, with $(K^2) = 9$
- to ruled surface with $(K^2) = 8$.

Flatness not enough, example 2

The next example is built on
M. Franciosi, R. Pardini, S. Rollenske (2017)

similar ones constructed and used by
Y. Lee, J. Park (2007)
J. Keum, Y. Lee, H. Park (2012)
E. Dias, C. Rito, G. Urzúa (2022)
J. Reyes, G. Urzúa (2022)
H. Park, J. Park, D. Shin (2023)

$(\mathbb{Z}/2)^2$-covers

$S = S(f, g, h)$: composite of $\mathbb{P}^2(\sqrt{fg}), \mathbb{P}^2(\sqrt{gh}), \mathbb{P}^2(\sqrt{hf})$.

Generic local structure: $(f, g, h) = (x, y, 1)$.
composite of $\mathbb{A}^2(\sqrt{x}), \mathbb{A}^2(\sqrt{y})$: smooth

Special $(f, g, h) = (x, y, x - y)$: $\mathbb{A}^2_{uv}/\frac{1}{4}(1, 1)$ with
$x = (u^2 + v^2)^2, y = (u^2 - v^2)^2, x - y = (2uv)^2$.
Check:

$$\sqrt{xy} = u^4 - v^4, \sqrt{x(x-y)} = 2uv(u^2+v^2), \sqrt{y(x-y)} = 2uv(u^2-v^2$$

Flatness not enough, example 3

Pick $f, g, h \in \mathbb{C}[x, y, z]$, homog. of degrees $3, 3, 1$.
$S = S(f, g, h)$: composite of $\mathbb{P}^2(\sqrt{fg}), \mathbb{P}^2(\sqrt{gh}), \mathbb{P}^2(\sqrt{hf})$.

$\mathbb{P}^2(\sqrt{fg})$ is a K3, so

$K_S \sim C$:=preimage of $L := (h = 0)$.

Corollary. For general f, g, h, the surface S is smooth, K_S is ample and $(K_S^2) = 1$, Godeaux surface.

Flatness not enough, example 4

Special case: S_0: when $f = g = h = 0$ has 2 points.

S_0 has 2 points $\mathbb{C}^2/\frac{1}{4}(1,1)$, both on C.

Resolve: get E_1, E_2 with $(E_i^2) = -4$ and

$$\overset{-4}{E_1} = \overset{-1}{C} = \overset{-4}{E_2}$$

We still have $K_{S_0'} \sim C$.
Contract C: get $S_0' \to T_0$ and T_0 is a K3 surface!

Lemma. Since S_0 has rational singularites, any flat deformation of S_0' contracts to a flat deformation of S_0.

Flatness not enough, example 5

Corollary. S_0 has 2 kinds of flat smoothings:

$$\left\{ \begin{array}{c} \text{K3 surface} \\ \text{blown up once} \end{array} \right\} \rightsquigarrow S_0 \leftsquigarrow \left\{ \begin{array}{c} \text{Godeaux surface} \\ K \text{ ample, } (K^2) = 1 \end{array} \right\}$$

The K3 surface can be non-algebraic.

What distinguishes the two sides?

Good deformation direction

Let $p : X \to (0 \in C)$ represent

$$S_0 \leftsquigarrow \left\{ \begin{array}{c} \text{Godeaux surface} \\ K \text{ ample, } (K^2) = 1 \end{array} \right\}.$$

Then $2K_{X/C}$ is a Cartier divisor.

Bad deformation direction

Let $q : Y \to (0 \in C)$ represent

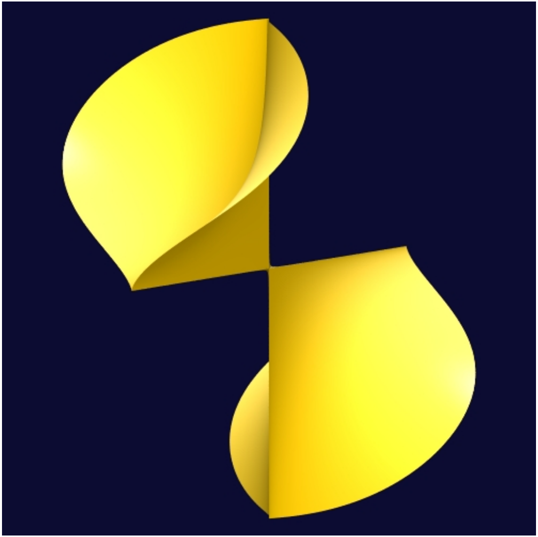
$$\left\{ \begin{array}{c} \text{K3 surface} \\ \text{blown up once} \end{array} \right\} \rightsquigarrow S_0$$

Then no multiple of $K_{Y/C}$ is a Cartier divisor.

However

- for infinitesimal deformations of cyclic quotients, the "$mK_{Y/C}$ is Cartier" condition is rather unpredictable (Altmann–Kollár, 2019).

Questions?



Families of algebraic varieties
Felix Klein Lecture # 3
János Kollár

Characterizations of stable families

Recall: Stable variety

- Deminormal (= at worst nodes in codim $1 + S_2$),
- semi-log-canonical singularities,
  (examples: quotient or cone over Fano or CY)
- projective and ω_X is ample.

Example from yesterday

A surface S_0 with $\mathbb{C}^2/\frac{1}{4}(1,1)$ singularities, ample K_{S_0}, and 2 kinds of flat smoothings:

$$\left\{ \begin{array}{c} \text{K3 surface} \\ \text{blown up once} \end{array} \right\} \rightsquigarrow S_0 \leftsquigarrow \left\{ \begin{array}{c} \text{Godeaux surface} \\ K \text{ ample, } (K^2) = 1 \end{array} \right\}$$

What distinguishes the two sides?

Recall: $\omega_X^{[m]} :=$ reflexive hull of $\omega_X^{\otimes m}$.

Theorem

Let $X \to S$ be a flat, proper morphisms with stable fibers, S reduced. Equivalent:

1. *The volume of the fibers $(K^n_{X_s})$ is locally constant.*
2. *The plurigenera $h^0(X_s, \omega^{[m]}_{X_s})$ are locally constant $\forall m$.*
3. *$\omega^{[m]}_{X/S}$ is flat and commutes with base change $\forall m$.*
4. *The $\omega^{[m]}_{X_s}$ are the fibers of $\omega^{[m]}_{X/S}$.*

Note: For version with "sufficienty divisible m" we need about the fibers only that:

S_2 (Serre's condition, eg. normal)
ω_{X_s} localy free in codim 1,
$\omega^{[m_s]}_{X_s}$ localy free and ample for some $m_s > 0$.

Clear:

(3) $\omega_{X/S}^{[m]}$ is flat and commutes with base change $\forall m$.

$\Rightarrow$

(4) The $\omega_{X_s}^{[m]}$ are the fibers of $\omega_{X/S}^{[m]}$.

(4) $\Rightarrow$ (2)

Assume (4): the $\omega_{X_s}^{[m]}$ form a flat family. $\Rightarrow$

(4') $\chi(X_s, \omega_{X_s}^{[m]})$ is locally constant.

Next: $h^0(X_s, \omega_{X_s}^{[m]}) = \chi(X_s, \omega_{X_s}^{[m]})$ if
either $m \gg 1$ (by Serre)
or $m \geq 2$ and stable fibers (by Ambro–Fujino vanishing)
(Kawamata–Viehweg not enough).

$m = 1$: Come back to this in Lecture 4.

(2) $\Rightarrow$ (3)

(2): $s \mapsto h^0(X_s, \omega_{X_s}^{[m]})$ are locally constant.

If $\omega_{X/S}^{[M]}$ is locally free, then $\omega_{X/S}^{[m+rM]} \cong \omega_{X/S}^{[m]} \otimes \left(\omega_{X/S}^{[M]}\right)^r$, so

Hilbert polynomial of $\omega_{X_s}^{[m]}$ is independent of s, so

(3): $\omega_{X/S}^{[m]}$ is flat and commutes with base change.

(2) $\Rightarrow$ (1)

(2): $s \mapsto h^0(X_s, \omega_{X_s}^{[m]})$ are locally constant.

The $\left(K_{X_s}^n\right)$ are the leading terms, so

(1): $s \mapsto \left(K_{X_s}^n\right)$ is also locally constant.

(1) $\Rightarrow$ (2) slide 1

Theorem

Let $X \to S$ be flat, projective, with S_2 fibers; S reduced. L reflexive rank 1 sheaf, locally free in codim 1 on each fiber. Assume that each $(L_s)^{**}$ is locally free and ample. Then

1. $s \mapsto \mathrm{volume}\big((L_s)^{**}\big)$ *is upper semicontinuous, and*
2. *locally constant iff L is locally free.*

Apply this to $\omega_{X/S}^{[m]}$ such that every $\omega_{X_s}^{[m]}$ is locally free.

Other m: Come back to this in Lecture 4.

Typical example (with divisors)

$X = (xy - u^2 + t^2v^2 = 0) \subset \mathbb{P}^3 \times \mathbb{A}^1_t$,
$D := (x = u - tv = 0) + (y = u + tv = 0)$.

$t \neq 0$: X_t smooth, D_t Cartier and $(D_t^2) = 0$.
$t = 0$: X_0 cone , $D_0 = (u = 0)$ is Cartier, and $(D_0^2) = 2$.

Add hyperplane class:

$$(H_0 + D_0)^2 = 8, \quad (H_t + D_t)^2 = 6.$$

Note: D not Cartier at $x = y = u = t = 0$.

(1) $\Rightarrow$ (2) slide 2

$n = 2$ case: here cokernel of $L \to L_0^{**}$ is 0 dimensional, so

$$\chi\big(X_0, (L_0^{**})^m\big) \geq \chi\big(X_g, (L_g^{**})^m\big) = \chi\big(X_g, L_g^m\big).$$

Riemann-Roch:

$$\tfrac{1}{2}(L_0^{**} \cdot L_0^{**})m^2 + b_0 m + \chi(X_0) \geq \tfrac{1}{2}(L_g \cdot L_g)m^2 + b_g m + \chi(X_g).$$

So $(L_0^{**} \cdot L_0^{**}) \geq (L_g \cdot L_g)$, and if $=$ then

$$b_0 m \geq b_g m \quad \forall m \in \mathbb{Z}.$$

So $b_0 = b_g$.

(1) $\Rightarrow$ (2) slide 3

$n \geq 3$ induction (nontrivial) reduces to special case:
L is locally free except at isolated points.

Local Grothendieck-Lefschetz: $x \in X_0 \subset X$ then

$\mathrm{Pic}(X \setminus \{x\}) \hookrightarrow \mathrm{Pic}(X_0 \setminus \{x\})$ if $\operatorname{depth}_x X_0 \geq 3$.

Problem: we have only $\operatorname{depth}_x X_0 \geq 2$.

Theorem. Still ok if $\operatorname{depth}_x X_0 \geq 2$ and $\dim X_0 \geq 3$.

– slc case (K, 2013)
– normal case (Bhatt - de Jong, 2014)
– general case (K, 2016)
– see Stacks Tag 0FB2

Aside: Fulger - K - Lehmann, 2016

X normal, proper, D big $\mathbb{R}$ divisor, E effective $\mathbb{R}$ divisor.
Equivalent:

- $H^0\big(X, \mathcal{O}_X \llcorner m(D-E) \lrcorner\big) = H^0\big(X, \mathcal{O}_X \llcorner mD \lrcorner\big)\ \forall m \geq 1$.
- $\mathrm{volume}(D-E) = \mathrm{volume}(D)$.

Answer to Question 3

Definition

A morphism $f : X \to S$ is stable iff

1. f *is flat, projective with stable fibers, and*
2. $\omega_{X/S}^{[m]}$ *is flat and commutes with base change* $\forall m$.

Thesis

Stable morphisms are the correct higher dimensional analogs of flat families of stable curves of genus ≥ 2.

- Warning: Problems in char p (see Lecture 6).

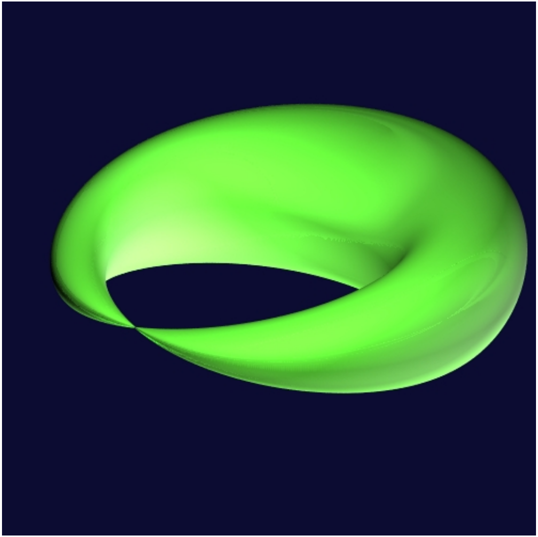
Terminology

- This is called KSB stable (= Kollár – Shepherd-Barron).
I am sure that this is the right notion.

- Later: pairs (X, Δ) where Δ is a $\mathbb{Q}$ or $\mathbb{R}$-divisor.
There are several versions, they agree over reduced bases.
These are called KSBA stable
(= Kollár – Shepherd-Barron – Alexeev).

- The distinction is not systematic.

Questions?



Families of stable varieties

Question. Fix a property $\mathcal{P}$ and $f : X \to S$ flat, proper.
Is $\mathcal{P}(S) := \{s \in S, X_s \text{ satisfies } \mathcal{P}\} \subset S$ open?

Yes:
— (geometrically) reduced or normal, or
— rational singularities (Elkik, 1978), or
— canonical singularities (Kawamata 1999, Kollár 2013).

No: semi-log-canonical or stable.

A bad example

A smooth curve of genus g,
L line bundle of degree $2g-2$.

$$C_L(A) := \operatorname{Spec} \oplus_m H^0(A, L^m) \text{ cone over } A, \text{ using } L$$

Claim. Class group at the vertex is $\operatorname{Pic}(A)/\langle L \rangle$.

Corollary. The canonical class is $\mathbb{Q}$-Cartier iff
$\omega_A \otimes L^{-1}$ is torsion.

Corollary. Over $\operatorname{Pic}^{2g-2}(A)$ we have a flat family of cones $C_L(A)$. The canonical class of the fibers is $\mathbb{Q}$-Cartier over a discrete but Zariski dense subset of $\operatorname{Pic}^{2g-2}(A)$.

Luckily: These are not log canonical.
For $g = 1$ this family does not exist.

Stability is representable

– $f : X \to S$: flat, proper, characteristic 0.
– fibers at worst nodal in codim 1.

Theorem

There is a monomorphism $i_S : S^{\mathrm{stable}} \to S$ such that, for every $g : T \to S$, the following are equiv.

1. *The pull-back $f_T : X_T \to T$ is stable.*
2. *g factors through i_S.*

Being slc is not open condition 1

Family of 3-folds in $\mathbb{P}^5_x \times \mathbb{A}^2_{st}$:

$$X := \left(rank \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 + sx_4 & x_2 + tx_5 & x_3 \end{pmatrix} \leq 1 \right).$$

Claim: the following are equivalent:

- X_{st} is semi-log-canonical (in fact klt)
- $3K_{X_{st}}$ is Cartier
- either $(s, t) = (0, 0)$ or $st \neq 0$.

Being slc is not open condition 2

Case 1: $st \neq 0$:

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 + sx_4 & x_2 + tx_5 & x_3 \end{pmatrix} \to \begin{pmatrix} x_0 & x_1 & x_2 \\ x_4 & x_5 & x_3 \end{pmatrix}$$

This is $\mathbb{P}^1 \times \mathbb{P}^2$, hence even smooth.

Being slc is not open condition 3

Case 2: $s = t = 0$:

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 + sx_4 & x_2 + tx_5 & x_3 \end{pmatrix} \to \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

This is the 2-cone over $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$.
The singularity is locally like $\mathbb{C}^3/\frac{1}{3}(1,1,0)$:
$\mathbb{Z}/3\mathbb{Z}$ acts with $(\epsilon, \epsilon, 1)$.

Being slc is not open condition 4

Case 3: $s = 0, t \neq 0$:

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 + sx_4 & x_2 + tx_5 & x_3 \end{pmatrix} \to \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_5 & x_3 \end{pmatrix}$$

This is the cone over $F_1 \hookrightarrow \mathbb{P}^4$.
F_1 is Fano but this is **not** the anticanonical embedding.
Here $-K_{F_1}$ is not proportional to hyperplane class.

Being stable is not open condition 5

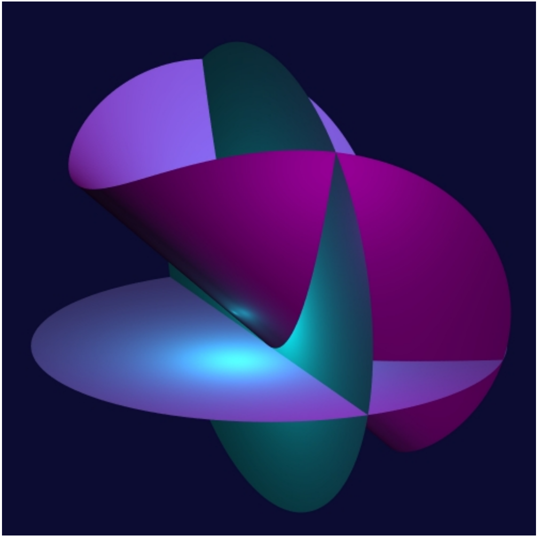
Let $Y \subset \mathbb{P}^6_x \times \mathbb{A}^1_{st}$ be the family of 3-folds

$$\sum_{i=0}^{6} x_i^m = 0 \text{ and } rank \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 + sx_4 & x_2 + tx_5 & x_3 \end{pmatrix} \leq 1.$$

Claim: for $m \geq 5$ the following are equivalent:

- Y_{st} is stable
- either $(s, t) = (0, 0)$ or $st \neq 0$.

Questions?



Main existence theorem (char 0)

Theorem

Fix positive n, v. There there is

1. *a DM-stack $\overline{\mathcal{M}}_{n,v}$ of stable morphisms whose fibers have dimension n and volume v, and*
2. *it has a projective coarse moduli space $\overline{\mathrm{M}}_{n,v}$.*

- modular properties as good as for $\overline{\mathcal{M}}_g$ and $\overline{\mathrm{M}}_g$,
- as a scheme, $\overline{\mathrm{M}}_{n,v}$ is much more complicated.

History of the proof

Surfaces:

– (existence) K–Shepherd-Barron (1988)
– (finite type) Alexeev (1993)
– (projectivity) K (1990)
– (local structure) arbitrarily bad, Vakil (2006)

Higher dimensions

– (existence) K (2011)
– (finite type) Karu (2000), Hacon–McKernan–Xu (2018)
– (projectivity) Fujino (2018), Kovács–Patakfalvi (2017)

Why is finite type hard?

$C = \cup_{i\in I} C_i$ stable curve of genus g. Then
$2g-2 = \deg \omega_C = \sum_{i\in I} \deg(\omega_C|_{C_i}) \geq |I|$.

$S = \cup_{i\in I} S_i$ stable surface. Then
$K_S|_{\bar{S}_i} \sim K_{\bar{S}_i} + D_i$ (D_i=nodal locus), so
$(K_S^2) = \sum_{i\in I}(K_{\bar{S}_i} + D_i)^2 \geq ??$

Here a multiple $m_i(K_{\bar{S}_i} + D_i)$ is Cartier, so we get that
$(K_S^2) \geq \sum_{i\in I} 1/m_i^2$.

Need to bound m_i from above to bound $|I|$.

This is not possible, but, we can bound all possible $(K_{\bar{S}_i} + D_i)^2$ from below.

A surface bound

Theorem (K., Alexeev-Liu, Liu-Shokurov)

If (S, D) *surface pair, log canonical,* $K_S + D$ *ample, then*

$$(K_S + D)^2 \geq \frac{1}{462}.$$

(Esser-Totaro, 2023) The extreme example can be realized as $X_{42} \subset \mathbb{P}^3(6, 11, 14, 21)$.

Higher dimensions:

- Ineffective bound (Hacon–McKernan–Xu, 2018), and
- smaller than $1/2^{2^n}$ (Esser-Totaro). ($n = \dim$).

Weak boundedness is easier

Theorem

Every irreducible component of $\overline{\mathrm{M}}_{n,v}$ is proper.

$\pi^\circ : U^\circ \to M^\circ$ universal family over open set.

Semi-stable reduction (Karu, Abramovich-Karu)
After finite cover $N^\circ \to M^\circ$ can compactify
$N^\circ \subset \bar{N}$, such that we have
$\bar{\pi} : \bar{U} \to \bar{N}$, flat with toroidal fibers.

Run MMP, we get (this is bit tricky, see next)
$\bar{\pi}^{\mathrm{can}} : \bar{U}^{\mathrm{can}} \to \bar{N}$ flat with stable fibers.

Corollary. $\bar{N} \to \overline{\mathrm{M}}_{n,v}$ has proper image, which is the closure of M°.

No stabilization functor

C reduced, nodal curve. There is a natural $C \mapsto C^{\mathrm{stab}}$ that works in flat families.

S reduced, snc surface. There is no $S \mapsto S^{\mathrm{stab}}$ that works in flat families.

Theorem (Kollár-Xu)

$\pi : X \to B$, flat, slc fibers, generic fiber of general type. B smooth and $K_{X/B}$ is $\mathbb{Q}$-Cartier. Then

1. *$\pi^{\mathrm{can}} : X^{\mathrm{can}} \to B$ is flat with stable fibers, but*
2. *$X_b \dashrightarrow (X^{\mathrm{can}})_b$ depends on $X \to B$, not just on X_b.*

Note: (2) happens even for irreducible, klt fibers.

Questions?