Families of algebraic varieties
Felix Klein Lectures

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Plan of the lectures

- History and examples, from Riemann to Mumford
- Moduli of varieties; main questions and definitions
- Characterizations of stable families
- Du Bois property and consequences
- K-flatness
- Difficulties in positive characteristic

— The lectures are mostly independent of each other.

— For details, see mainly the books

Singularities of the minimal model program, CUP, 2013

Families of varieties of general type, CUP, 2023

Families of algebraic varieties
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Du Bois singularities and consequences

When to turn to Du Bois singularities?

Thesis

If you have singularities that are not rational or CM, but close to it, Du Bois singularities may give the answer.

Example. $X \to S$ flat. When is $\omega_{X/S}$ flat?
Classical answer: if fibers are CM (easy proof).
New answer: if fibers are Du Bois.
(Will give longer, roundabout proof.)

Non-example: cones over $C \times D$, slide 1

Let C, D be a smooth, projective curves of genus ≥ 2, and L, M ample line bundles of degrees d, e. Let

$X_{L,M} := \operatorname{Spec} \bigoplus_{m \in \mathbb{Z}} H^0\big(C \times D, (L \boxtimes M)^m\big)$ be the cone over $C \times D$ with ample line bundle $L \boxtimes M$.

Then $\omega_{X_{L,M}}$ is the sheaf corresponding to $\bigoplus_{m \in \mathbb{Z}} H^0\big(C \times D, (\omega_C \boxtimes \omega_D) \otimes (L \boxtimes M)^m\big)$.

Examples. For $g - 1 < d \leq 2g - 2$ and suitable M

1. the $X_{L,M}$ form a flat family over $\operatorname{Pic}^d(C)$, but the $\omega_{X_{L,M}}$ are not flat; or
2. the $X_{L,M}$ do not form a flat family over $\operatorname{Pic}^d(C)$, but the $\omega_{X_{L,M}}$ are flat.

Non-example: cones over $C \times D$, slide 2

Key property: $h^0(C, L^m)$ and $h^0(C, \omega_C \otimes L^{-m})$ vary with L only for $m = 1$.

So, the only summands that vary with L are

- $H^0(C, L) \otimes H^0(D, M)$ in $\mathcal{O}_{X_{L,M}}$, and
- $H^0(C, \omega_C \otimes L^{-1}) \otimes H^0(D, \omega_D \otimes M^{-1})$ in $\omega_{X_{L,M}}$.

Therefore:

- $X_{L,M}$ **not** flat over $\mathrm{Pic}^d(C)$ iff $H^0(D, M) \neq 0$, and
- $\omega_{X_{L,M}}$ **not** flat over $\mathrm{Pic}^d(C)$ iff $H^0(D, \omega_D \otimes M^{-1}) \neq 0$.

When is ω flat?

X proper of dimension n, L ample. Then

- $\omega_X =$ sheaf of $\oplus_m H^0(X, \omega_X \otimes L^m)$, and
- $H^0(X, \omega_X \otimes L^m)$ is dual to $H^n(X, L^{-m})$.

Corollary. $g : X \to S$ projective, relative dim n. Then $\omega_{X/S}$ is flat and commutes with base changes iff $R^n g_* L^{-m}$ is free for $m \gg 1$.

Principles:

- $\omega_X^\bullet$ is encoded in the $H^i(X, L^{-1})$ for all L ample.
- If need help, ask Sándor Kovács.

Detour: cyclic covers 1

For $s \in H^0(X, L^{[m]})$ we have $\pi : X\left[\sqrt[m]{s}\right] \to X$ as

- $\mathrm{Spec}_X\left(\mathcal{O}_X \oplus L^{[-1]} \oplus \cdots \oplus L^{[1-m]}\right)$, or as
- $(s = 0) \subset \mathrm{Spec}_X \oplus_{r \geq 0} L^{[r]}$.

Note that

- $\pi_* \omega_{X[\sqrt[m]{s}]} \cong \omega_X \oplus \omega_X [\otimes] L \oplus \cdots \oplus \omega_X [\otimes] L^{[m-1]}$.

Thus, if $L = \omega_X$ then

- $\pi_* \omega_{X[\sqrt[m]{s}]} \cong \omega_X \oplus \cdots \oplus \omega_X^{[m]}$.

Detour: cyclic covers 2

If L ample, then

- L^{-1} is direct summand of $\pi_* \mathcal{O}_{X[\sqrt[m]{s}]}$, so
- $H^i(X, L^{-1})$ is direct summand of $H^i\left(X\left[\sqrt[m]{s}\right], \mathcal{O}_{X[\sqrt[m]{s}]}\right)$.

If $\omega_X = L$ ample, then

- $\omega_X^{[r]}$ are direct summands of $\pi_* \omega_{X[\sqrt[m]{s}]}$, so
- $H^i(X, \omega_X^{[r]})$ are direct summands of $H^i\left(X\left[\sqrt[m]{s}\right], \omega_{X[\sqrt[m]{s}]}\right)$.

When is ω flat?

Let $\mathcal{S}$ be a class of singularities, closed under

- $X \mapsto X \times \mathbb{A}^1$, and
- general hyperplane sections $X \mapsto H \cap X$,
- so general cyclic covers with invertible L.

$\mathrm{Flat}_n(\mathcal{S}) :=$ all $g : X \to B$
flat, projective, relative dim n, fibers in $\mathcal{S}$.

Corollary. For $\mathcal{S}$ equivalent:

- $R^n g_* L^{-m}$ is locally free for all ample L, and for all $(g : X \to B) \in \mathrm{Flat}(\mathcal{S})$.
- $R^n g_* \mathcal{O}_X$ is locally free for all $(g : X \to B) \in \mathrm{Flat}(\mathcal{S})$.

When is $H^i(X, \mathcal{O}_X)$ flat?

Cohomology and base change

Let A be Artinian with residue field k, and $g : X_A \to \operatorname{Spec} A$ flat, proper.

Equivalent:

- the $H^i(X_A, \mathcal{O}_{X_A})$ are free A-modules.
- $H^i(X_A, \mathcal{O}_{X_A}) \twoheadrightarrow H^i(X_k, \mathcal{O}_{X_k})$.

Illustration for $A = k[\epsilon]$:

$$\begin{array}{ccccc} H^i(X_k, \mathcal{O}_{X_k}) & \stackrel{\epsilon}{\to} & H^i(X_A, \mathcal{O}_{X_A}) & \to & H^i(X_k, \mathcal{O}_{X_k}) \\ H^{i+1}(X_k, \mathcal{O}_{X_k}) & \stackrel{\epsilon}{\to} & H^{i+1}(X_A, \mathcal{O}_{X_A}) & \to & H^{i+1}(X_k, \mathcal{O}_{X_k}) \end{array}$$

Du Bois singularities 1

Global defn (incorrect):
$H^i(X, \mathbb{C}) \twoheadrightarrow H^i(X, \mathcal{O}_X)$ if X proper and DB.

Theorem (Du Bois–Jarraud, 1974) If X_k is DB then
$H^i(X_A, \mathcal{O}_{X_A}) \twoheadrightarrow H^i(X_k, \mathcal{O}_{X_k})$.
(A Artinian with residue field k)

Proof.

$$\begin{array}{ccc} H^i(X_A, \mathcal{O}_{X_A}) & \rightarrow & H^i(X_k, \mathcal{O}_{X_k}) \\ \uparrow & & \uparrow \\ H^i(X_A, \mathbb{C}) & \rightarrow & H^i(X_k, \mathbb{C}) \end{array}$$

Recall: semi-log-canonical =
singularities we have on limits of canonical models.

- Deminormal:= X only nodes in codimension 1 and S_2
  (so ω_X line bundle in codim 1),
- $\omega_X^{[m]}$ is locally free for some $m > 0$ (with section σ^m),
- Three equivalent versions:

— Using resolution I: $K_Y \sim p^* K_X + \text{(effective)} - E$,
where E = reduced exceptional divisor.

— Using resolution II: there is $p^* \omega_X^{[r]} \to \omega_Y^{[r]}(rE) \quad \forall r \geq 0$.

— Using local volume: $\int_X \sigma \wedge \bar{\sigma}$ has only

logarithmic growth:= $\left| \int_X |g|^\epsilon \cdot \sigma \wedge \bar{\sigma} \right| < \infty$,

for every g vanishing on $\operatorname{Sing} X$ and $\epsilon > 0$.

Du Bois singularities 2

Theorem (Kollár-Kovács, 2010)

Semi-log-canonical is Du Bois.

(More generally, (X, Δ) slc, then any union of log canonical centers is Du Bois. Kollár-Kovács, 2010, 2020).

Corollary. Let $g : X \to S$ be flat, fibers slc. Then

$\omega_{X/S}$ is flat over S and commutes with base change.

What about the other $\omega_{X/S}^{[r]}$?

Theorem

$X \to S$ flat with slc fibers, S reduced and $\omega_{X/S}^{[m]}$ is locally free for some $m > 0$. Then all $\omega_{X/S}^{[r]}$ are flat and commute with base change.

Proof. Assume $S = C$ is a smooth curve and $\omega_{X/C}^{[m]}$ is free.

Take $\pi : X[\sqrt[m]{s}] \to X$.

Reid's lemma: $X[\sqrt[m]{s}]$ is log canonical

Elkik, ...: $\omega_{X[\sqrt[m]{s}]/C}$ has S_2 fibers

Recall: The $\omega_{X/C}^{[r]}$ are direct summands of $\pi_* \omega_{X[\sqrt[m]{s}]/C}$.

So fibers of $\omega_{X/C}^{[r]}$ agree with $\omega_{X_c}^{[r]}$.

Back to definition of stable morphisms 1

The definition of ‘stable morphism’ included:

$(*)$ The $\omega^{[r]}_{X/B}$ are flat and commute with base change.

Thesis

In defining stable morphisms:

1. *over smooth curves, we proved* $(*)$*,*
2. *over reduced bases,* $(*)$ *works out, and*
3. *over general bases, we have to require* $(*)$*.*

Back to definition of stable morphisms 2

Theorem. (Altmann–Kollár, 2019) For many cyclic quotients $S_0 = \mathbb{C}^2/\frac{1}{n}(1,q)$ there are flat deformations $S \to \operatorname{Spec} A$ for $A := \mathbb{C}[\epsilon]$, such that,

- $\omega_{S/A}^{[n]}$ is free, but
- $\omega_{S/A}^{[r]}$ is not flat if $r \not\equiv 0, 1 \mod n$.

Corollary The assumption $(*)$:

"the $\omega_{X/B}^{[r]}$ are flat over B"

needs to be added by hand for families of surfaces.

Stability in codimension 3

Theorem (Kollár–Kovács, 2023)

Stability is automatic in codimension ≥ 3.

That is:

Let $f : X \to B$ be flat and finite type, such that

- fibers are semi-log-canonical, and
- locally stable in codim ≤ 2 (in each fiber).

Then locally stable everywhere.

Note. Can allow non-flatness in codim ≥ 3.

Question. Is this true for pairs (X, Δ)?

Key Theorem

Key Theorem. Let $f : X \to B$ be finite type, such that

- flat with Du Bois fibers in codim ≤ 2 (in each fiber),
- fibers have Du Bois (partial) normalization.

Then $\omega_{X/B}$ is flat and commutes with base change.

Surprise. We can not show that $\mathcal{O}_X$ is flat.

Lemma. $\pi : \bar{Y} \to Y$ isom in codim ≤ 1. Then
$\pi_* \omega_{\bar{Y}} \cong \omega_Y$.

Proof of Key Theorem, slide 1

As before, one ingredient is the following:

Claim. Let A be Artinian with residue field k, and

$g : X_A \to \operatorname{Spec} A$, proper, pure dim n.

Then $H^n(X_A, \mathcal{O}_{X_A})$ is a free A-module if

- g is flat in codimension ≤ 2, and
- $H^i(X_k, \mathbb{C}) \twoheadrightarrow H^i(X_k, \mathcal{O}_{X_k})$ for $i = n, n{-}1$.

Illustration for $A = k[\epsilon]$:

$$\begin{array}{ccccc} H^{n-1}(X_k, \mathcal{O}_{X_k}) & \xrightarrow{\epsilon} & H^{n-1}(X_A, \mathcal{O}_{X_A}) & \to & H^{n-1}(X_k, \mathcal{O}_{X_k}) \\ H^{n}(X_k, \mathcal{O}_{X_k}) & \xrightarrow{\epsilon} & H^{n}(X_A, \mathcal{O}_{X_A}) & \to & H^{n}(X_k, \mathcal{O}_{X_k}) \end{array}$$

Proof of Key Theorem, slide 2

Y: pure dim n, (embedded points allowed)
$\tau : \bar{Y} \to Y$ partial normalization.

Claim $H^i(Y, \mathbb{C}) \twoheadrightarrow H^i(Y, \mathcal{O}_Y)$ for $i = n, n-1$ if

- $\bar{Y}$ is Du Bois,
- $\tau : \bar{Y} \to Y$ is a homeomorphism, and
- $\tau : \bar{Y} \to Y$ is isomorphism in codimension ≤ 2.

Proof.

$$\begin{array}{ccc} H^i(Y, \mathbb{C}) & \to & H^i(Y, \mathcal{O}_Y) \\ \downarrow & & \downarrow \\ H^i(\bar{Y}, \mathbb{C}) & \to & H^i(\bar{Y}, \mathcal{O}_{\bar{Y}}) \end{array}$$

Does this ever happen?

Slc version of Key Theorem

(A Artinian case)
Let $f : X \to \operatorname{Spec} A$ be finite type, such that

- slc (partial) normalization $\bar{X}_k \to X_k$, and
- locally stable in codim ≤ 2 (in each fiber).

Then $\omega_{X/B}$ is flat and commutes with base change.

Thus, if $\omega_{\bar{X}_k}$ is locally free (and X is S_2), then:

- $\omega_{X/A}$ is flat and locally free (!),
- $\mathcal{O}_X$ is flat, and
- all the $\omega^r_{X/A}$ are flat and locally free.

- So $f : X \to \operatorname{Spec} A$ is locally stable.

How to make ω locally free?

Lemma. If $\omega_U^{[m]}$ is free, take cyclic cover

$$\pi : \bar{U} := \mathrm{Spec}_U(\mathcal{O}_U \oplus \omega_U \oplus \cdots \oplus \omega_U^{[m-1]}) \to U.$$

Then $\omega_{\bar{U}}$ is free.

Proof. We know that $\pi_* \omega_{\bar{U}} \cong \oplus_{i=0}^{m-1} Hom_U(\omega_U^{[i]}, \omega_U)$

and we have 1 as a section of the $i = 1$ summand

$Hom_U(\omega_U, \omega_U) \cong \mathcal{O}_U$.

Stability in codimension 3 – proof 1

Induction + working locally, assume that:

- $\omega_{X_k}^{[m]}$ is free, and
- $X \to \operatorname{Spec} A$ is stable on $U := X \setminus \{x\}$.

So $\omega_{X/A}^{[-m]} \in$ kernel of $\operatorname{Pic}(U) \to \operatorname{Pic}(U_k)$.

Fact: this kernel is a vector space, so divisible.

Thus there is a unique line bundle L_U on U (with push-forward L on X) such that

$L_{U_k} \sim \mathcal{O}_{U_k}$ and $(\omega_{X/A} \otimes L)^{[-m]} \cong \mathcal{O}_X$.

Note. L flat/A iff free.

Cyclic cover $\pi : Y := \operatorname{Spec}_X \oplus_{i=0}^{m-1} (\omega_{X/A} \otimes L)^{[i]} \to X$.

Stability in codimension 3 – proof 2

$\pi : Y := \mathrm{Spec}_X \oplus_{i=0}^{m-1} (\omega_{X/A} \otimes L)^{[i]} \to X.$

Over U_k we have $\mathrm{Spec}_{U_k} \oplus_{i=0}^{m-1} \omega_{U_k}^{[i]}$, so

$\bar{Y}_k \cong \mathrm{Spec}_{X_k} \oplus_{i=0}^{m-1} \omega_{X_k}^{[i]} \to Y_k$ is partial normalization.

Note: Y_k could have embedded points over x!

By slc version of Key Theorem: $\omega_{Y/A}$ and $\mathcal{O}_Y$ are both flat.

$\pi_* \omega_{Y/A}$ has a summand $Hom_U \big(\omega_X \otimes L, \omega_X \big) \cong L^{[-1]}$,

so $L^{[-1]}$ flat/A, and $L \cong \mathcal{O}_X$.

So $\pi_* \mathcal{O}_Y \cong \oplus_{i=0}^{m-1} \omega_{X/A}^{[i]}$, and all summands are flat.

Definition of Du Bois

There is a filtered complex $\underline{\Omega}^{\bullet}_X$ that computes the mixed Hodge structure on $H^{\bullet}(X, \mathbb{C})$, hence

$H^i(X, \mathbb{C}) \twoheadrightarrow H^i(X, \underline{\Omega}^{\circ}_X)$ for any proper X.

$$\begin{array}{ccc} H^i(X,\mathbb{C}) & \twoheadrightarrow & H^i(X,\underline{\Omega}^{\circ}_X) \\ \downarrow & \nearrow & \\ H^i(X,\mathcal{O}_X) & & \end{array}$$

Correct definition of Du Bois:

$\mathcal{O}_X \to \underline{\Omega}^{\circ}_X$ is a quasi-isomorphism.

Local cohomology lifting

A Artinian with residue field k
$g : X \to \operatorname{Spec} A$ finite type (not assumed flat)

Theorem (Kollár-Kovács, 2020) Assume

1. either char$= 0$ and X_k is Du Bois,
2. or char> 0 and X_k is F-pure.

Then for every $x \in X$ and i:

$$H^i_x(X, \mathcal{O}_X) \twoheadrightarrow H^i_x(X_k, \mathcal{O}_{X_k}).$$

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K-flatness

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K-flatness

Why K?

K-flatness

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Originally had C-flat for Cayley, but needed new notion.

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K-flatness

Why K?

Originally had C-flat for Cayley, but needed new notion.

K= Cayley.

Moduli of pairs (X, Δ)

Objects. Replace K_X by $K_X + \sum a_i D_i$.

Families. $g : (X, \Delta = \sum a_i D_i) \to S$ such that

- fibers are stable pairs,
- $K_{X/S} + \sum a_i D_i$ is $\mathbb{Q}$-Cartier, and
- the D_i are $\boxed{????}$.

For 30 years we had a theory where a basic definition was not known.

Answer given finally in (K. 2019).

For ???? flatness is too much

Example (Hassett, 1993)

Smooth quadric degenerates to quadric cone:

$$(xy + z^2 - t^2u^2 = 0) \subset \mathbb{P}^3 \times \mathbb{A}^1.$$

$D_0 = L_0 + \frac{1}{2}(L_0' + L_0'')$ (lines through vertex)

$D_t = L_t + \frac{1}{2}(L_t' + L_t'')$ (where $L_t' \cap L_t'' = \emptyset$).

Note: $\chi(L_0' + L_0'') = 1$, but $\chi(L_t' + L_t'') = 2$.

(Can get irreducible examples too.)

Coefficients $> \frac{1}{2}$ – slide 1

Theorem (Kollár, 2014)

Let $(X, \sum_{i \in I} a_i D_i) \to S$ *be stable, with* S *reduced. Assume that* $a_i > \frac{1}{2}$*. Then, for every* $J \subset I$,

$$\cup_{i \in J} D_i \to S$$

is flat with reduced fibers.

Coefficients $> \frac{1}{2}$ – slide 2

Corollary. If $a_i > \frac{1}{2}$, we can handle the moduli problem as

- $X \to S$ is flat, and
- $\boxed{????} :=$ flat, so the D_i are in $\mathrm{Hilb}(X/S)$.

However, this is not possible if $a_i \leq \frac{1}{2}$.

Mumford divisors — 1

$g : X \to S$ projective of pure relative dim n.

Definition. $D \subset X$ a relative Mumford divisor iff

$(*)$ g smooth and D is Cartier at η_s,

for all $s \in S$ and all generic points $\eta_s \in D_s$.

Corollary. D_s defined as a divisor on X_s:
Cartier at generic points, then take closure.

Warning.
D_s is **not** the scheme-theoretic fiber.
The later can have embedded subschemes.

Mumford divisors — 2

Thesis

Over reduced bases, the correct higher dimensional analogs of flat families of pointed stable curves are:
Stable families $g : (X, \sum a_i D_i) \to S$,
where the $D_i \to S$ *are* ***Mumford****.*

Mumford divisors over $k[\epsilon]$— example

Local version. $\mathrm{Pic}\big(\mathbb{A}^2_{k[\epsilon]} \setminus \{(0,0)\}\big)$ is infinite dimensional.

Example: $I_n := (x^2, xy^n + \epsilon, \epsilon x) \subset k[x, y, \epsilon]$. Note that

$k[x, y, \epsilon]/I_n \cong k[x, y]/(x^2)$, but

$k[x, y, \epsilon]/(I_n, \epsilon) \cong k[x, y]/(x^2, xy^n)$ with

torsion ideal: $\langle x, xy, \ldots, xy^{n-1}\rangle$.

Projective version.

The space of Mumford divisors $D \subset \mathbb{P}^2_{k[\epsilon]}$ such that
$D_k = (\text{line}) + (\text{embedded points})$ is
infinite dimensional.

K-flatness — first definition

Let $g : D \to S$ be projective, pure relative dimension $n-1$.
Assume that at generic points of each fiber

- g is flat, and
- embedding dimension of fiber $\leq n$.

Assume first: S local with infinite residue field.

Definition

$g : D \to S$ is K-flat iff all images $D \to \mathbb{P}^n_S$ are flat over S.*

In general: the previous holds for all
localization + residue field extension.

Thesis

The correct higher dimensional analogs of flat families of pointed stable curves are:
Stable families $g : (X, \sum a_i D_i) \to S$,
where the $D_i \to S$ *are K-flat.*

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The correct higher dimensional analogs of flat families of pointed stable curves are:
Stable families $g : (X, \sum a_i D_i) \to S$,
where the $D_i \to S$ *are* *K-flat*.

However, although

K-flatness is a surprisingly good property,
there could be other possibilities.

Example – plane curves over $k[\epsilon]$

Start with $C := (f(x, y) = 0)$.

Flat defs: $f(x, y) = \psi(x, y)\epsilon$ where $\psi \in k[x, y]$

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Flat defs of $C \setminus \{(0,0)\}$:

$$(*) \quad f(x,y) = \psi(x,y)\epsilon, \quad z = \phi(x,y)\epsilon$$

where ψ, ϕ regular on $C \setminus \{(0,0)\}$.

Theorem. $(*)$ is K-flat iff ψ is regular on C and

- $f_x\phi, f_y\phi$ are regular on C.

Example. Monomial curve $(x^c = y^a)$ or $t \mapsto (t^a, t^c)$.

- becomes: $t^{ac-a}\phi(t), t^{ac-c}\phi(t) \in k[t^a, t^c]$.

Get $(a-1)(c-1)$-dim family of K-flat but non-flat defs.

Cayley coordinates

called:

Cayley form in Hodge–Pedoe

Zugeordnete Form by van der Waerden

coordonnées de Chow in French

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“cette horreur de coordonnées de Chow”

Serre letter to Grothendieck, 1956

Cayley flatness – 1

$C \subset \mathbb{P}^3$ a curve.

Cayley hypersurface (Cayley, 1860):

$$\mathrm{Ca}(C) := \{L \in \mathrm{Grass}(1,3) : L \cap C \neq \emptyset\}.$$

Note that

$\mathrm{Ca}(C) =$
$\cup_{p \in \mathbb{P}^3}(\text{lines through } p \text{ that meet } C) =$
$\cup_{p \in \mathbb{P}^3}(\text{image of projection of } C \text{ from } p).$

Cayley flatness – 2

$Z \subset \mathbb{P}^N$ of pure dimension $n-1$

Cayley hypersurface:

$$\mathrm{Ca}(Z) := \{L \in \mathrm{Grass}(N-n, N) : L \cap Z \neq \emptyset\}.$$

Set $G := \mathrm{Grass}(N-n-1, N)$ (=projection centers).

Note that

$\mathrm{Ca}(Z) =$
$\cup_{M \in G}(L \text{ through } M \text{ that meet } Z) =$
$\cup_{M \in G}(\text{image of projection of } Z \text{ from } M) =$
$\cup(\text{images of all projections } Z \to \mathbb{P}^n).$

Cayley flatness – Basic Theorems (Kollár, 2019)

Theorem 1. One can extend the definition of Cayley hypersurface to families $D \subset \mathbb{P}^N_S$, assuming that

- pure relative dimension $n-1$,
- g is flat at generic points of each fiber, and
- fibers have embedding dimension $\leq n$ at generic points.

Theorem 2. Assume S local with infinite residue field. The following are equivalent:

- $\mathrm{Ca}(D) \to S$ is flat over S.
- the images of all* projections $D \to \mathbb{P}^n_S$ are flat over S.
- the images of general projections $D \to \mathbb{P}^n_S$ are flat$/S$.

This is called C-flatness.

Cayley flatness – five versions – 1

C-flatness: all linear projections $D \to \mathbb{P}^n_S$.

Stable C-flatness: all linear projections composed with Veronese embeddings $D \to \mathbb{P}^n_S$.

K-flatness: all morphisms $D \to \mathbb{P}^n_S$.

Local K-flatness: all local morphisms $D \supset D^\circ \to \mathbb{A}^n_S$.

Formal K-flatness: all morphisms after all completions $\widehat{D} \to \widehat{\mathbb{A}}^n_S$.

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Conjecture. They are all equivalent.

Cayley flatness – five versions – 2

C-flatness: all linear projections $D \to \mathbb{P}^n_S$.

Stable C-flatness: all linear projections composed with Veronese embeddings $D \to \mathbb{P}^n_S$.

K-flatness: all morphisms $D \to \mathbb{P}^n_S$.

Local K-flatness: all local morphisms $D \supset D^\circ \to \mathbb{A}^n_S$.

Formal K-flatness: all morphisms after completion $\widehat{D} \to \widehat{\mathbb{A}}^n_S$.

Theorem. The red ones are equivalent.

Cayley flatness – five versions – 3

Some subtle points:

- A morphism $D_s \to \mathbb{P}^n_s$ may not extend to $D \to \mathbb{P}^n_S$.

Cayley flatness – five versions – 3

Some subtle points:

- A morphism $D_s \to \mathbb{P}^n_s$ may not extend to $D \to \mathbb{P}^n_S$.
- There is no Noether normalization in families:

$U \to S$ affine of dim 1, may not be a finite morphism
$U \to \mathbb{A}^1_S$.

K-flatness — good properties

$g : X \to S$ projective of pure relative dim n, and
$D \subset X$ a relative Mumford divisor.

- flat $\Rightarrow$ K-flat.
- $D \to S$ is K-flat $\Leftrightarrow$ $D_A \to A$ are K-flat $\forall$ Artinian A.
- if g is smooth, then flat $\Leftrightarrow$ K-flat.
- if D_s are normal, then flat $\Leftrightarrow$ K-flat.
- if S is reduced, then Mumford $\Leftrightarrow$ K-flat.

K-flatness — good properties

$g : X \to S$ projective of pure relative dim n, and
$D \subset X$ a relative Mumford divisor.

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- if g is smooth, then flat $\Leftrightarrow$ K-flat.
- if D_s are normal, then flat $\Leftrightarrow$ K-flat.
- if S is reduced, then Mumford $\Leftrightarrow$ K-flat.

- if D_i are K-flat then $\sum D_i$ is K-flat.
- D is K-flat $\Leftrightarrow$ mD is K-flat (if $p \nmid m$).
- preserved by linear equivalence.

K-flatness — Bertini theorem

$g : X \to S$ projective of pure relative dim $n \geq 3$, and
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Theorem (Up-down Bertini theorem)

D is K-flat iff $D|_H$ is K-flat.

Main reason: $\mathrm{Pic}\big(\mathbb{A}^n_A \setminus \{0\}\big) = 0$ for $n \geq 3$, A Artinian.
(see Lecture 6).

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Corollary

K-flatness is about divisors on surfaces.

Computing projections — 1

Note:
$\{\text{roots of } f(t)\} = \{\text{eigenvectors of } t\cdot \text{ on } k[t]/(f)\}$

Claim. M finite $R[t]$-algebra (or module).
Assume M is free over R: $M = \oplus_{i=1}^{n} e_i R$.
Write $t \cdot e_i = \sum r_{ij} e_j$ with $r_{ij} \in R$.

Then, the equation of projection to $\operatorname{Spec} R[t]$ is
$\det\bigl(1_n t - (r_{ij})\bigr) = 0$.

Computing projections — 2

We project to $\operatorname{Spec} A[[u,v]]$ with A Artinian.

May assume: $M := \mathcal{O}_D$ is

- finite over $A[[u]]$, and
- free over $A((u))$ of rank say n.

So our equation is:

$\det\big(1_n v - (r_{ij}(u))\big) = 0$, where $r_{ij}(u) \in A((u))$.

The projection is

- flat over $A[[u]]$ $\Leftrightarrow$ the equation is in $A[[u,v]]$.

Proof of: K-flat = stable C-flat: slide 1

Using Up-down Bertini theorem
reduces to dimension 1.

Main advantage of dim 1: can ignore high terms:

For $f(u), g(u) \in \mathbb{C}((u))$, we have

$fg \in \mathbb{C}[[u]] \Leftrightarrow (f + u^M)(g + u^M) \in \mathbb{C}[[u]]$ for $M \gg 1$.

2 dim example:

$$\frac{1}{u - \sin v} \cdot (u - \sin v + v^M) \text{ never in } \mathbb{C}[[u, v]]$$

Proof of: K-flat = stable C-flat: slide 2

Maps of $\mathbb{A}^2_{uv}$ to $\mathbb{A}^1_u$ used for:

C-flatness: $(u, v) \mapsto (u, au + bv)$.

K-flatness: $(u, v) \mapsto \big(u, \phi(u, v)\big)$, where ϕ power series,

C-flatness with dth Veronese: $(u, v) \mapsto \big(u, h(u, v)\big)$,
where $\deg h \leq d$.

Lemma. Given a holomorphic $\phi(u, v)$ with matrix $\big(r_{ij}(u)\big)$ there is a polynomial $\phi'(u, v)$ with matrix $\big(r'_{ij}(u)\big)$, s.t.

$$r_{ij}(u) \equiv r'_{ij}(u) \mod (u^M) \qquad (M \gg 1)$$

Corollary.

$$\det\big(1_n v - (r_{ij})\big) \in A[[u, v]] \;\Leftrightarrow\; \det\big(1_n v - (r'_{ij})\big) \in A[[u, v]]$$

Families of algebraic varieties
Felix Klein Lecture # 6
János Kollár

Positive characteristic

New phenomena

- Jumps in plurigenera, hence non-flatness of families of canonical models.
- There are too many $\mathbb{Q}$-Cartier divisors.

Open question from Lecture 3

Version 1. Are the $h^0(X, \omega_X^m)$ deformation invariant?

Version 2. Is there a natural transformation

$$\left\{ \begin{array}{c} \text{smooth families} \\ \text{of varieties of} \\ \text{general type} \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} \text{flat families of} \\ \text{canonical models} \end{array} \right\}?$$

In char p:

- Open for $\dim \geq 3$.
- Fail for pairs (with mild singularities).

Jump in plurigenera 1

$g : X \to C$ smooth, projective. Fiberwise canonical models:

$$X_c \mapsto X_c^{\mathrm{can}} := \mathrm{Proj} \oplus H^0(X_c, \omega_{X_c}^m).$$

Canonical models form flat family iff, for $m \gg 1$,

$$c \mapsto P_m(X_c) := h^0(X_c, \omega_{X_c}^m) \text{ is constant.}$$

Many known examples where finitely many P_m jump:
Katsura–Ueno (1985): elliptic surfaces,
Suh (2008): ample K

First example with infinitely many P_m jump:
Brivio (2020): elliptic surfaces (S, Δ).

Jump in plurigenera – plan of example

- jump of $H^0(S_c, \mathcal{O}_{S_c}(m))$ for ruled surfaces,
- from S_c to 3-folds with $K_{X_c} + \Delta_c$ semi-ample and big,
- $X^{\mathrm{can}} := \mathrm{Proj}_C \oplus g_* \mathcal{O}_X\left(mK_{X/C} + \llcorner m\Delta \lrcorner\right) \to C$ exists,
- $(X_0)^{\mathrm{can}} \to (X^{\mathrm{can}})_0$ is a birational homeomorphism, but purely inseparable over a single curve $\mathbb{P}^1 \subset (X^{\mathrm{can}})_0$.

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- $(X_0)^{\mathrm{can}} \to (X^{\mathrm{can}})_0$ is a birational homeomorphism, but purely inseparable over a single curve $\mathbb{P}^1 \subset (X^{\mathrm{can}})_0$.
- Unexpected: X_0 and $(X_0)^{\mathrm{can}}$ lift to char 0, so Kodaira vanishing holds on them.

$\mathbb{P}^1$-bundles on E – slide 1

Fix E elliptic curve and $\mathcal{O}_E \to F \to \mathcal{O}_E$ non-split. Get

$\pi : S \to E$: $\mathbb{P}^1$-bundle with section $D \cong E$.

Note: $(D^2) = 0$ and $K_S \sim -2D$.

Claim. $h^0(S, \mathcal{O}_S(mD)) = 1$ if char$= 0$, and
$h^0(S, \mathcal{O}_S(pD)) = 2$ if char$= p > 0$.

Proof. Let $C \in |mD|$ be irreducible, reduced curve.
Then $(C \cdot K_S) = -2(C \cdot D) = 0$. So $p_a(C) = 1$.
Projection: $\pi_C : C \to E$ finite, so C elliptic.

Key: $\pi_C^* S$ has 2 sections: C and D. So $\pi_C^* F$ is split.

$\Leftrightarrow \pi_C^* : H^1(E, \mathcal{O}_E) \to H^1(C, \mathcal{O}_C)$ is zero map.

$\mathbb{P}^1$-bundles on E – slide 2

Char 0: $\frac{1}{\deg \pi}\mathrm{Trace}$ splits $\mathcal{O}_E \to \mathcal{O}_C$, so

$$H^1(E, \mathcal{O}_E) \hookrightarrow H^1(C, \mathcal{O}_C).$$

Char p: there is a $C \to E$ of degree p such that

$$H^1(E, \mathcal{O}_E) \to H^1(C, \mathcal{O}_C) \text{ is zero map.}$$

(iff $\mathrm{Pic}(E) \to \mathrm{Pic}(C)$ is inseparable)

Corollary. $|pD| : S \to \mathbb{P}^1$ is an elliptic surface,

(with a wild fiber pD).

$\mathbb{P}^1$-bundles on E – slide 3

- Choose $\{S_t : t \in \mathbb{A}^1\}$ such that $S_t \cong S$ for $t \neq 0$ and $S_0 \cong E \times \mathbb{P}^1$.
  We have $\{D_t \subset S_t\}$ such that
  $\dim |pD_t| = 1$ for $t \neq 0$ and $\dim |pD_0| = p$.
- Set $\Delta := \frac{1}{np}$(sum of $3n$ general members of $|pD|$).
- Then $K_{S_t} + \Delta_t \sim_{\mathbb{Q}} D_t$.

Conclusion. All sufficiently large (log) plurigenera of (S_t, Δ_t) jump at $t = 0$.

McKernan trick

Start with (S, Δ_S) and $X := \mathrm{Proj}_S(\mathcal{O}_S + \mathcal{O}_S(1))$

Pull-back: Δ_X and $E \subset X$ negative section.

$H :=$ sum of (at least 3) general positive sections.

Claim. $(X, H + E + \Delta_X)$ is (log) general type, and

$$H^0\big(S, \mathcal{O}_S(mK_S + \llcorner m\Delta_S \lrcorner)\big)$$
$$||$$
$$H^0\big(E, \mathcal{O}_E(mK_E + \llcorner m\Delta_E \lrcorner)\big)$$
$$\downarrow \text{(direct summand)}$$
$$H^0\big(X, \mathcal{O}_X(mK_X + m(E + H) + \llcorner m\Delta_X \lrcorner)\big).$$

Proof: $0 \to \omega_X \to \omega_X(E) \to \omega_E \to 0$ and $\mathbb{C}^\times$-action.

Jump in plurigenera – conclusion

Set $\Theta := H_t + E_t + \Delta_t$. We have

$\{(X_t, \Theta_t) : t \in \mathbb{A}^1\}$ such that

$(X_t, \Theta_t) \to \left(X_t^{\mathrm{can}}, \Theta_t^{\mathrm{can}}\right)$ isom off E_t, and induces

$|D_0| : S_0 \cong E_0 \to \mathbb{P}^1$ for $t = 0$, and

$|pD_t| : S_t \cong E_t \to \mathbb{P}^1$ for $t \neq 0$.

Thus the flat limit of $|pD_t|$ as $t \to 0$ is

$$S_0 \cong E_0 \xrightarrow{|D_0|} \mathbb{P}^1 \xrightarrow{Frob} \mathbb{P}^1.$$

Note: only X_0 lifts to char 0.

Lauritzen–Kovács–Totaro–Bernasconi type examples

Homogeneous spaces $X = X_t$ degerate to
X_0:= cone over a hyperplane section.

In some cases with *non-reduced* stabilizer:

— Kodaira vanishing fails on X,
— X_0 not CM at vertex, and
— does not lift to char 0.

Strongest examples: $\pi : Y \to C$ such that

- K_Y is π-ample,
- Y_c smooth for $c \neq 0$,
- $\bar{Y}_0$ has canonical singularities,
- $\bar{Y}_0 \to Y_0$ is isomorphism, except at a single point,
- dimension $\sim$ twice the characteristic.

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Problems occur even in the interior of the moduli space!

Open questions

Main Question. How to define stable families in char p?

Question 2. Surfaces in chars 2,3,5? For char ≥ 7:
Patakfalvi (2017), Arvidsson–Bernasconi–Patakfalvi (2023)

Question 3. Plurigenera of smooth 3-folds? (without Δ)

Question 4. Semi-stable reduction? (even for surfaces!)

Difficulty 2:
There are too many $\mathbb{Q}$-Cartier divisors

Picard group over $k[\epsilon]$

$U_A \to \operatorname{Spec} A$ flat over $A = k[\epsilon]$

$$0 \to \mathcal{O}_{U_0} \xrightarrow{\epsilon} \mathcal{O}_{U_A} \to \mathcal{O}_{U_0} \to 0$$

$$0 \to \mathcal{O}_{U_0} \xrightarrow{1+\epsilon\cdot} \mathcal{O}_{U_A}^{\times} \to \mathcal{O}_{U_0}^{\times} \to 1$$

$$H^1(U_0, \mathcal{O}_{U_0}) \to \operatorname{Pic}(U_A) \to \operatorname{Pic}(U_0)$$

$(x, X_A) \to \operatorname{Spec} A$ isolated singularity, $U_A := X_A \setminus \{x\}$:

$$H^2_x(X_0, \mathcal{O}_{X_0}) = H^1(U_0, \mathcal{O}_{U_0}) \to \operatorname{Pic}^{\mathrm{loc}}(X_A) \to \operatorname{Pic}^{\mathrm{loc}}(X_0)$$

Local Picard group over $k[\epsilon]$

$$H^2_x(X_0, \mathcal{O}_{X_0}) \to \mathrm{Pic}^{\mathrm{loc}}(X_A) \to \mathrm{Pic}^{\mathrm{loc}}(X_0)$$

Claim. $H^2_x(X_0, \mathcal{O}_{X_0})$ is

- k^∞ if $\dim X_0 = 2$,
- 0 if $\dim X_0 \geq 3$ and CM,
- k^{finite} if $\dim X_0 \geq 3$.

Corollary. If $L_0 \in \mathrm{Pic}^{\mathrm{loc}}(X_0)$ is torsion, then:

- unique torsion lifting if char $k = 0$, and
- all liftings torsion if char $k > 0$.

K_X in local Picard group — typical example

Example. Take $X \subset \mathbb{P}^5 \times \mathbb{A}^1$ such that
$X_0 =$ cone over deg 4 rational normal curve, and
$X_t = \mathbb{P}^1 \times \mathbb{P}^1$ (embedded by $\mathcal{O}(2,1)$).

Then, for $X_n \subset \mathbb{P}^5 \times \operatorname{Spec} k[t]/(t^{n+1})$,

- $2K_{X_0}$ is Cartier,
- K_X is not $\mathbb{Q}$-Cartier,
- K_{X_1} is not $\mathbb{Q}$-Cartier if char $k = 0$, and
- K_{X_n} is $\mathbb{Q}$-Cartier $\forall n$ if char $k > 0$.

Aside: K_X in local Picard group – Lee–Nakayama (2018)

$K_{X/C}$ is the only possible $\mathbb{Q}$-Cartier lifting of K_{X_0}

Theorem. $X \to (0, C)$ flat, X_0 is slc, char=0.
D: $\mathbb{Q}$-Cartier divisor such that $D_0 \sim K_{X_0}$.

Then $D \sim K_{X/C} + (\text{Cartier divisor})$.

Moduli consequences of too many $\mathbb{Q}$-Cartier divisors

Points on $\mathbb{P}^1$ – slide 1

Objects over $\bar{k}$*:* $\mathbb{P}^1$ plus n unordered points.

Objects over k*:* Smooth, geometrically rational curve, plus a reduced subscheme of length n.

Families: $\mathbb{P}^1$–bundle $P_S \to S$ plus $D \subset P_S$, a $\mathbb{Q}$-Cartier divisor of degree n over S.

Bases: Reduced only.

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Theorem. The categorical moduli space is $\mathrm{M}^{\mathbb{Q}}_{0,n} \cong \operatorname{Spec} k$.

Comments

Note that we assume:

Families: $\mathbb{P}^1$–bundle $P_S \to S$ plus $D \subset P_S$, a Q-Cartier divisor of degree n over S.

Insisting on Cartier would fix the problem here.

However, in higher dimensions we do have non-Cartier limits, so $\mathbb{Q}$-Cartier is the strongest we can require.

Points on $\mathbb{P}^1$ – Descending families

Start with:

- B smooth curve and $D \subset \mathbb{P}^1 \times B$ degree n Cartier divisor,
- $\pi : B \to B'$ birational with B' higher cusps only (for example $k[t^m, t^{m+1}]$)

New family:

- $\mathbb{P}^1 \times B'$ and $D' := (\pi, 1_P)_* D$.

Claim. D' is $\mathbb{Q}$-Cartier if char$= p > 0$.

Proof: For $q > m^2$ factors as $\mathrm{Frob}_q : B \xrightarrow{\pi} B' \xrightarrow{\tau} B$.

So $q \cdot D' = q \cdot (\pi, 1_P)_* D = (\tau, 1_P)^* D$.

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Aside. If char$= 0$, then D' is not* $\mathbb{Q}$-Cartier

Corollary. Fix a smooth curve B and $h : B \to \mathrm{M}_{0,n}^{\mathbb{Q}}$.
Let B' be any curve with higher cusps and
normalization $\pi : B \to B'$.
Then h factors as

$$h : B \xrightarrow{\pi} B' \xrightarrow{h'} \mathrm{M}_{0,n}^{\mathbb{Q}}$$

Exercise. Fix Z. If every $B \to Z$ factors through every $B \to B'$, then $Z = \operatorname{Spec} k$.

Complement. One can do the same with
2-dimensional *senimormal* bases, using:

$$k[x] + (y^q - x)k[x, y] \subset k[x, y],$$

which is senimormal, with normalization $k[x, y]$.

Proposal to solve the $\mathbb{Q}$-Cartier problem

We have to impose the additional

Assumption: Let $\pi : X \to S$ be stable and $D \subset X$ a $\mathbb{Q}$-Cartier relative Mumford divisor.
Pick $x \in X$ and $s = \pi(x)$. Then:

If $m_x \cdot D_s$ is Cartier at x, then $m_x \cdot D$ is Cartier at x.

Comment. If $p = char$, then $p^c m_x D$ is Cartier for some c.

Warning. May need adjusting if D_s has multiplicities.