Some Existence Results for Fluid-Structure Interaction Problems

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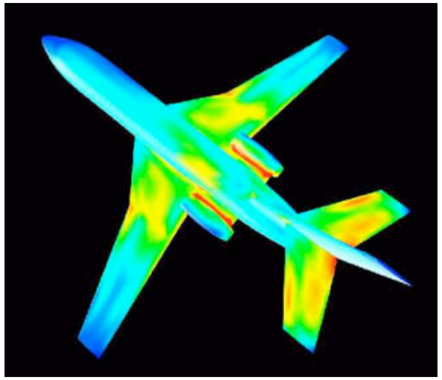
Motivations

Exemples

- Aerodynamic: Aeronautics, automobil;
- Hydrodynamic: boat, physiological flows;



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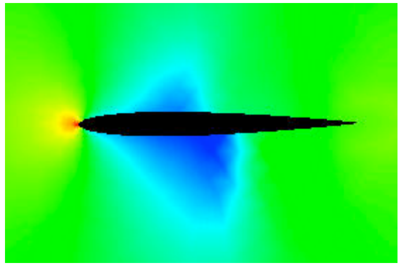
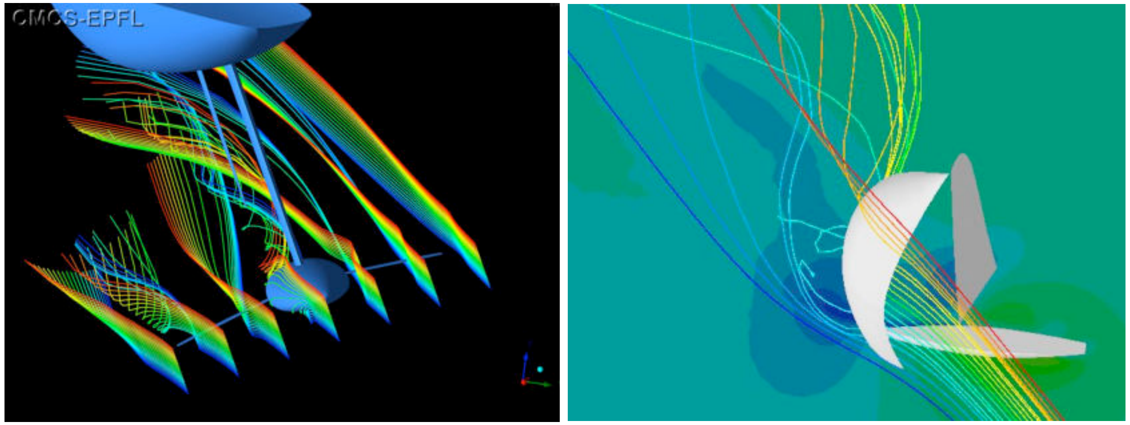
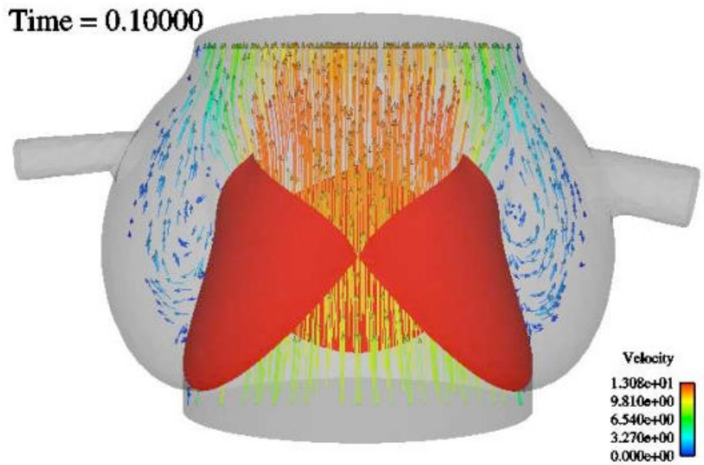




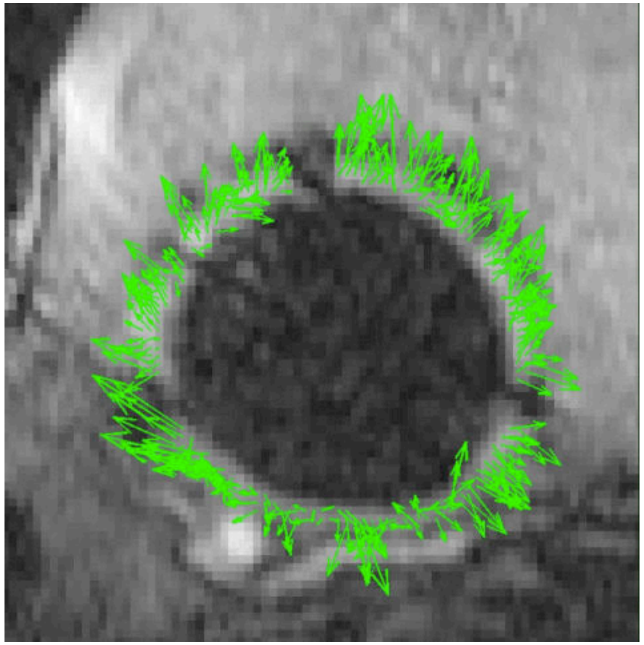
EPFL, A.Quarteroni

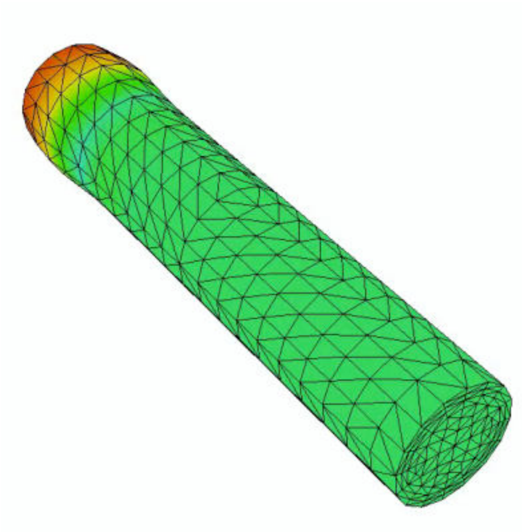
Inria, Cardiac Valve, M. Astorino,

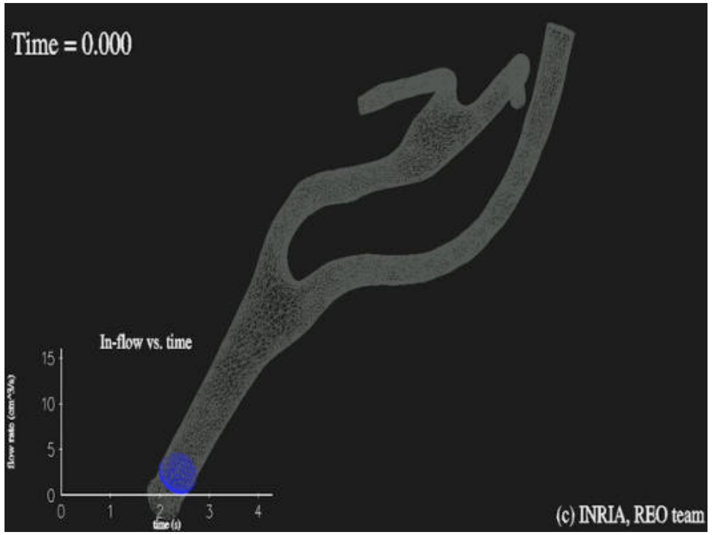


Motivations

- Modelling physiological flows such as blood flow in large arteries







Steps:

Physiolocigal problem $\Longrightarrow$ Mathematical and numerical problem $\Longrightarrow$ Numerical simulation

Motivations

Mathemathical analysis provides

— A mathematical framework (energy spaces, existence, uniqueness...) ;

— A framework to design efficient schemes ;

— Insight on the limit of the model.

Fluid-Structure Interaction

- Fluid: Newtonian, viscous, incompressible

  $\Longrightarrow$ Navier-Stokes equations

- Structure: Elastic media, large displacements

  $\Longrightarrow$ 3D elasticity, plate/shell models, rigid body...

- Coupling conditions at the interface:

  $\Longrightarrow$ Equality of the velocities

  $\Longrightarrow$ Equality of the strain tensor

$\Longrightarrow$ Energy balance at the interface

Mathematical Model

We consider a $2D$ container whose boundary is made of a $1D$ elastic rod or beam.

Fluid Domain:

$$\Omega_\eta(t) := \{(x,y) \in \mathbb{R}^2\,,\ x \in (0,L)\,,\ y \in (0, 1+\eta(x,t))\}\,.$$

Fluid:

$$\begin{cases} \rho^f \partial_t \mathbf{u} + \rho^f (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega_\eta(t), \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega_\eta(t), \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0 & \text{in } \Omega_{\eta_0}, \end{cases}$$

Structure:

$$\begin{cases} \rho_s \partial_{tt}\eta - \delta\partial_{xxtt}\eta + \alpha\,\partial_x^4\eta - \beta\partial_{xx}\eta - \gamma\,\partial_{xxt}\eta = (T_f)_2(\mathbf{u},p,\eta) \text{ in } (0,L), \\ \eta(0,\cdot) = \eta_0,\ \partial_t\eta(0,\cdot) = \eta_1, \end{cases}$$

Boundary conditions:

- $\mathbf{u}(t,\cdot) = \mathbf{0} \quad$ on Γ_0.

Mathematical Model

Coupling conditions:

- Domain:

$$\Omega_\eta(t) = \phi(\eta)(\Omega_f)$$

- Equality of the velocities:

$$\mathbf{u}(t,x,1+\eta(t,x)) = (0, \partial_t \eta(t,x))^T, x \in (0,L).$$

- Force:

$$\int_0^L T_f \cdot \bar{\mathbf{v}} = \int_{\Sigma(t)} (-2\nu D(\mathbf{u}) \cdot \mathbf{n}_t + p\mathbf{n}_t) \cdot \mathbf{v} \qquad \forall \mathbf{v},$$

with $D(\mathbf{u}) = (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)/2$ and $\bar{\mathbf{v}}(t,x) = \mathbf{v}(t,x,1+\eta(t,x))$, $x \in (0,L)$.

Fluid boundary conditions

- Enclosed cavity

$$\mathbf{u} = \mathbf{0}, \quad \text{on } \Gamma_{in} \cup \Gamma_{out},$$

- Inflow-out flow boundary conditions

$$\boldsymbol{\sigma}_f(\mathbf{u}, p) \cdot \mathbf{n} = -p_{in}\mathbf{n}, \quad \text{on } \Gamma_{in},$$

$$\boldsymbol{\sigma}_f(\mathbf{u}, p) \cdot \mathbf{n} = -p_{out}\mathbf{n}, \quad \text{on } \Gamma_{out},$$

where $\boldsymbol{\sigma}_f(\mathbf{u}, p) = -p\boldsymbol{I} + 2\nu D(\mathbf{u})$ denotes the fluid stress tensor, $D(\mathbf{u})$ stands for the fluid strain tensor, $D(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$, and $\mathbf{n}$ denote the exterior unit normal to the considered boundary.

Note that many other boundary conditions could be imposed for instance

$$\boldsymbol{\sigma}_f(\mathbf{u}, p) \cdot \mathbf{n} \cdot \mathbf{n} = -p_{in}, \mathbf{u} \cdot \boldsymbol{\tau} = 0, \quad \text{on } \Gamma_{in},$$

$$\boldsymbol{\sigma}_f(\mathbf{u}, p) \cdot \mathbf{n} \cdot \mathbf{n} = -p_{out}, \mathbf{u} \cdot \boldsymbol{\tau} = 0, \quad \text{on } \Gamma_{out},$$

Remarks

- We have formally

$$(2D(\mathbf{u}) \cdot \mathbf{n}_t)_2 = (\nabla \mathbf{u} \cdot \mathbf{n}_t)_2, \quad \text{on } \partial\Omega_\eta(t) \setminus \Gamma_0.$$

and a Korn equatlity

$$\int_{\Omega_\eta(t)} |D(\mathbf{u})|^2 = \int_{\Omega_\eta(t)} |\nabla \mathbf{u}|^2.$$

- When considering homogeneous Dirichlet boundary conditions or periodic boundary conditions, the equalities of the velocities and the incompressibility imply

$$\int_\omega \partial_t \eta = 0.$$

This condition traduces the preservation of the volume of the fluid cavity.

- The pressure p is not defined up to an additive constant but is uniquelly defined. When considering homogeneous Dirichlet boundary conditions or periodic boundary conditions, its average is the Lagrange multiplier associated to the volume preserving constrain.

At least formally

$$u_1(x, 1+\eta(x)) = 0,$$

thus

$$\partial_x u_1(x, 1+\eta(x)) + \partial_x \eta(x) \partial_y u_1(x, 1+\eta(x)) = 0$$

But, due to the fluid incompressibility we obtain:

$$-\partial_y u_2(x, 1+\eta(x)) + \partial_x \eta(x) \partial_y u_1(x, 1+\eta(x)) = 0$$

Since

$$(\nabla \boldsymbol{u})^T = \begin{pmatrix} \partial_x u_1 & \partial_x u_2 \\ \partial_y u_1 & \partial_y u_2 \end{pmatrix}$$

and $\mathbf{n}_t = \frac{1}{\sqrt{1+(\partial_x \eta)^2}}(-\partial_x \eta, 1)^T$ we obtain that

$$((\nabla \boldsymbol{u})^T \cdot \mathbf{n}_t)_2 = \frac{1}{\sqrt{1+(\partial_x \eta)^2}} (\partial_y u_2(x, 1+\eta(x)) - \partial_x \eta(x) \partial_y u_1(x, 1+\eta(x))) = 0.$$

We then deduce that

$$\int_{\Omega_\eta(t)} (\nabla \boldsymbol{u})^T : \nabla \boldsymbol{u} = 0.$$

Even when considering homogeneous Dirichlet boundary conditions or periodic boundary conditions the pressure is uniquely defined. Indeed, the incompressibility condition together with boundary conditions imply:

$$\int_0^L \partial_t \eta = 0\,, \quad \forall\, t > 0\,.$$

Consequently,

$$\int_0^L (T_f)_2(\boldsymbol{u}, p, \eta) = 0.$$

The pressure can be then decomposed, for instance, as

$$p = p_0 + c,$$

where, one can choose to impose

$$\int_{\Omega_\eta(t)} p_0 = 0,$$

and where c satisfies

$$c(t) = \frac{1}{L}\int_0^L (T_f)_2(\boldsymbol{u}, p_0, \eta) = \frac{1}{L}\int_0^L e_2 \cdot (\sigma(\boldsymbol{u}, p_0))(x, 1+\eta(x,t), t)(-\partial_x \eta(x,t)\, e_1 + e_2).$$

Mathematical Analysis

Difficulties:

- Lack of energy estimates $\Longrightarrow$ look for smooth solutions;
- Geometrical non linearities:
  - lack of regularity of the structure displacement $\Longrightarrow$ look for smooth solutions;
  - work in non cylindrical time-space domain;
- Non linear coupled problem $\Longrightarrow$ need to obtain compactness;
- Hyperbolic-parabolic coupling $\Longrightarrow$ depending on the proof scheme, need to add additional viscosity;
- Added mass effect;
- How to deal with contact?
- How to obtain global existence of solution?

Energy estimates

- Enclosed cavity, x-periodic boundary conditions, or modified Neumann boundary conditions

$$\boldsymbol{\sigma}_f(\mathbf{u}, p + \rho_f \frac{|\mathbf{u}|^2}{2}) \cdot \mathbf{n} = -p_{in}\mathbf{n}, \quad \text{on } \Gamma_{in},$$

$$\boldsymbol{\sigma}_f(\mathbf{u}, p + \rho_f \frac{|\mathbf{u}|^2}{2}) \cdot \mathbf{n} = -p_{out}\mathbf{n}, \quad \text{on } \Gamma_{out},$$

$\Longrightarrow$ Energy estimates;

- Neumann boundary conditions $\Longrightarrow$ no energy estimates due to the kinetic energy fluxes at the outlets:

$$\int_{\Gamma_{in} \cup \Gamma_{out}} \rho_f \frac{|\mathbf{u}|^2}{2} \mathbf{u} \cdot \mathbf{n}$$

Energy Estimates

$$\rho^f \int_{\Omega_\eta(t)} \partial_t \mathbf{u}\cdot\mathbf{u} + \rho^f \int_{\Omega_\eta(t)} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u} + 2\nu \int_{\Omega_\eta(t)} |D(\mathbf{u})|^2 + \frac{\rho_s}{2}\frac{d}{dt}\int_0^L (\partial_t\eta)^2$$
$$+\delta\frac{d}{dt}\int_0^L (\partial_x\partial_t\eta)^2 + \frac{\alpha}{2}\frac{d}{dt}\int_0^L (\partial_{xx}\eta)^2 + \frac{\beta}{2}\frac{d}{dt}\int_0^L (\partial_x\eta)^2 + \gamma\int_0^L (\partial_x\partial_t\eta)^2 = \int_{\Omega_\eta(t)} \mathbf{f}\cdot\mathbf{u} + \int_0^L g\,\partial_t\eta.$$

But

$$\int_{\Omega_\eta(t)} \partial_t \mathbf{u}\cdot\mathbf{u} + \int_{\Omega_\eta(t)} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u} = \frac{1}{2}\frac{d}{dt}\int_{\Omega_\eta(t)} |\mathbf{u}|^2.$$

$\Longrightarrow$ Energy estimates:

$$\mathbf{u} \in L^\infty(0,T;L^2_\sharp(\Omega_\eta(t))), \quad D(\mathbf{u}) \in L^2(0,T;L^2_\sharp(\Omega_\eta(t))),$$

and

$$\eta \in W^{1,\infty}(0,T;L^2_\sharp(0,L)) \cap L^2(0,T;H^1_\sharp(0,L)),$$

for $\alpha > 0$

$$\eta \in L^2(0,T;H^2_\sharp(0,L)),$$

for $\gamma > 0$

$$\partial_t\eta \in L^2(0,T;H^1_\sharp(0,L)),$$

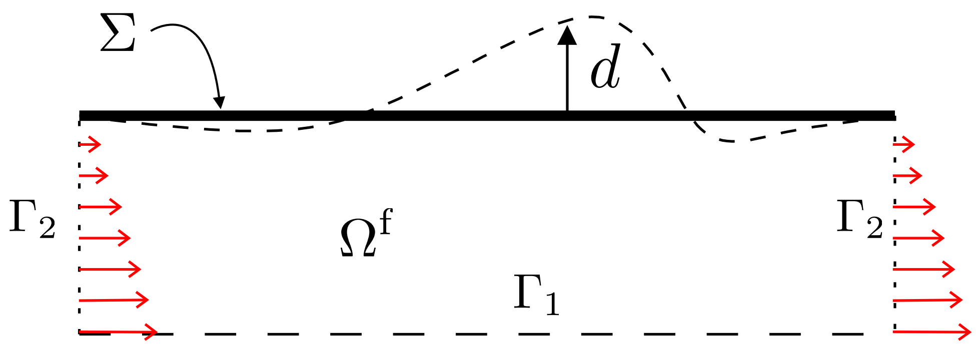
for $\delta > 0$

$$\partial_t\eta \in L^\infty(0,T;H^1_\sharp(0,L)).$$

Weak Formulation

$$\begin{aligned}
\rho_f \int_0^t \int_{\Omega_\eta(s)} \partial_t \mathbf{u} \cdot \boldsymbol{\phi} + \mu \int_0^t \int_{\Omega_\eta(s)} \nabla \mathbf{u} : \nabla \boldsymbol{\phi} + \rho_f \int_0^t \int_{\Omega_\eta(s)} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \boldsymbol{\phi} \\
+ \rho_s \int_0^t \int_0^L \partial_{tt} \eta \, b + \alpha \int_0^t \int_0^L \partial_{xx} \partial_t \eta \, \partial_{xx} b \\
+ \gamma \int_0^t \int_0^L \partial_x \partial_t \eta \, \partial_x b + \beta \int_0^t \int_0^L \partial_x \eta \, \partial_x b \\
= \int_0^t \int_{\Omega_\eta(s)} \mathbf{f} \cdot \boldsymbol{\phi} + \int_0^t \int_0^L g \; b, \\
\forall (\boldsymbol{\phi}, b) \text{ such that } \operatorname{div}(\boldsymbol{\phi}) = 0, \text{ in } \Omega_\eta(t), \\
\boldsymbol{\phi}(t, x, 1 + \eta(t, x)) = (0, b(t, x))^T \text{ on } (0, L).
\end{aligned}$$

The added mass: a simplified model



(Causin, JFG, Nobile, 2005)

- Solid: string model (infinitesimal displacements)

$$\rho^{\mathrm{s}} \partial_{tt} d + L d = p_{|\Sigma}, \quad \text{in} \quad \Sigma,$$

- Fluid: fixed fluid domain, no viscosity/convective terms

$$\left\{ \begin{aligned} \rho^{\mathrm{f}} \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla} p &= 0, && \text{in} \quad \Omega^{\mathrm{f}} \\ \operatorname{div} \boldsymbol{u} &= 0, && \text{in} \quad \Omega^{\mathrm{f}} \\ \boldsymbol{u} \cdot \boldsymbol{n} &= \dot{d}, && \text{on} \quad \Sigma \\ \boldsymbol{u} \cdot \boldsymbol{n} &= 0, && \text{on} \quad \Gamma_1 \\ p &= 0, && \text{on} \quad \Gamma_2 \end{aligned} \right. \quad \underset{\text{div}}{\Longrightarrow} \quad \left\{ \begin{aligned} -\Delta p &= 0, && \text{in} \quad \Omega^{\mathrm{f}} \\ \frac{\partial p}{\partial \boldsymbol{n}} &= -\rho^{\mathrm{f}} \frac{\partial \boldsymbol{u}}{\partial t} \cdot \boldsymbol{n} = -\rho^{\mathrm{f}} \partial_{tt} d, && \text{on} \quad \Sigma \\ \frac{\partial p}{\partial \boldsymbol{n}} &= 0, && \text{on} \quad \Gamma_1 \\ p &= 0 && \text{on} \quad \Gamma_2 \end{aligned} \right.$$

The added mass operator

Fluid: $$\begin{cases} -\Delta p = 0, & \text{in} \quad \Omega^{\mathrm{f}} \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = -\rho^{\mathrm{f}} \partial_{tt} d, & \text{on} \quad \Sigma \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = 0, & \text{on} \quad \Gamma_1 \\ p = 0 & \text{on} \quad \Gamma_2 \end{cases}$$

Solid: $$\rho^{\mathrm{s}} \partial_{tt} d + L d = p_{|\Sigma}, \quad \text{in} \quad \Sigma,$$

Steklov-Poincaré operator

The operator $\mathcal{M}_{\mathrm{A}} : H^{-\frac{1}{2}}(\Sigma) \to H^{\frac{1}{2}}(\Sigma)$ defined as: for each $g \in H^{-\frac{1}{2}}(\Sigma)$ we set $\mathcal{M}_{\mathrm{A}}(g) \stackrel{\text{def}}{=} q_{|\Gamma^{\mathrm{w}}}$, where $q \in H^1(\Omega^{\mathrm{f}})$ solves

$$\begin{cases} -\Delta q = 0, & \text{in} \quad \Omega^{\mathrm{f}} \\ \dfrac{\partial q}{\partial \boldsymbol{n}} = g, & \text{on} \quad \Sigma \\ \dfrac{\partial q}{\partial \boldsymbol{n}} = 0, & \text{on} \quad \Gamma_1 \\ q = 0, & \text{on} \quad \Gamma_2 \end{cases}$$

is a linear, compact, positive and self-adjoint operator in $L^2(\Sigma)$.

✓ From this definition, we have $$\boxed{p_{|\Sigma} = \mathcal{M}_{\mathrm{A}}(-\rho^{\mathrm{f}} \partial_{tt} d) = -\rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}} \partial_{tt} d}$$

The added mass effect

$$
\text{Fluid: } \begin{cases} -\Delta p = 0, & \text{in} \quad \Omega^{\mathrm{f}} \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = -\rho^{\mathrm{f}} \partial_{tt} d, & \text{on} \quad \Sigma \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = 0, & \text{on} \quad \Gamma_1 \\ p = 0 & \text{on} \quad \Gamma_2 \end{cases} \qquad \text{Solid: } \rho^{\mathrm{s}} \partial_{tt} d + L d = p_{|\Sigma}, \quad \text{in} \quad \Sigma, \tag{1}
$$

$$
\boxed{p_{|\Sigma} = -\rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}} \partial_{tt} d}
$$

$$
\left(\rho^{\mathrm{s}} \varepsilon + \rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}}\right) \partial_{tt} d + L d = 0, \quad \text{in} \quad \Sigma \tag{2}
$$

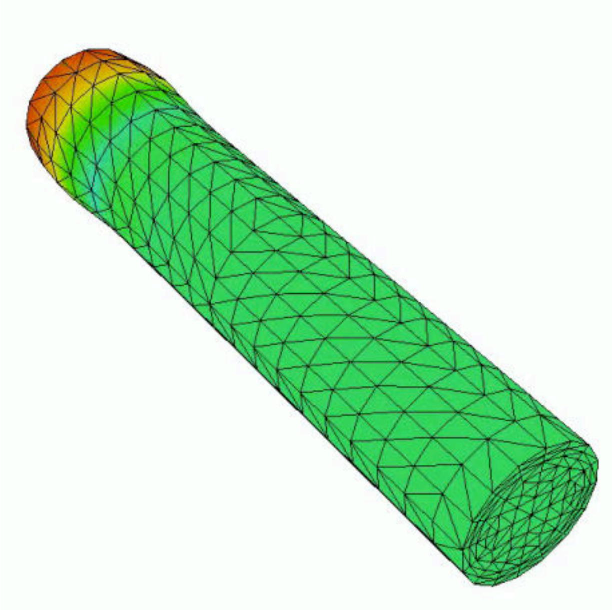
Remark:

- This equation looks like a structure equation, except for the extra “mass” term
- The **fluid-structure coupling** can be condensed into an **extra mass** action on the structure (hence the terminology “added-mass effect”)

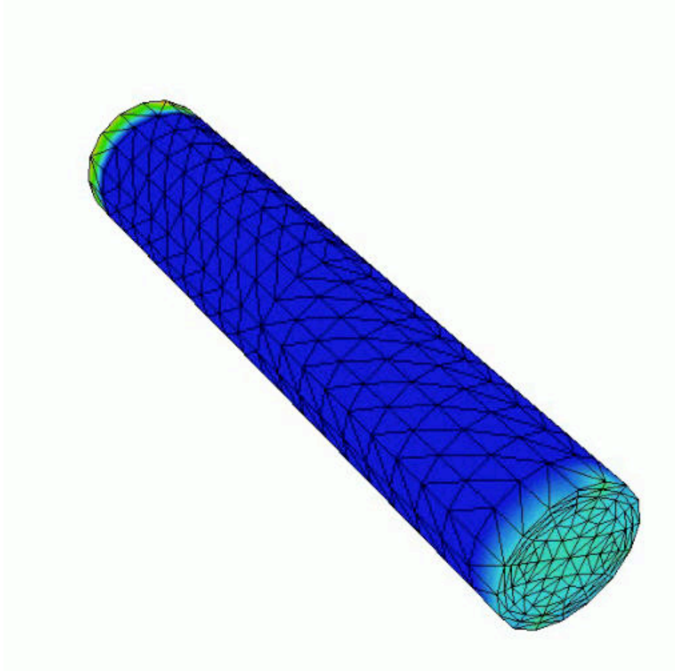
- If we try to perform a naive fixed point argument (Fluid+Dirichlet, Structure+Neumann)to prove existence of solution then it works only in the case where the mass of the structure is large enough compared to the one of the fluid.

Numerical evidence

Implicit coupling



Explicit coupling



- Instabilities disappear when the solid density is (artificially) increased
- Instabilities are independent of the time step
- The instability is sensitive to the length of the domain

We consider a $2D$ container whose boundary is made of a $1D$ elastic rod or beam.

Fluid Domain:

$$\Omega_\eta(t) := \{(x,y) \in \mathbb{R}^2\,,\; x \in (0,L)\,,\; y \in (0, 1+\eta(x,t))\}\,.$$

Fluid:

$$\begin{cases} \rho^f \partial_t \mathbf{u} + \rho^f (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega_\eta(t), \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega_\eta(t), \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0 & \text{in } \Omega_{\eta_0}, \end{cases}$$

Structure:

$$\begin{cases} \rho_s \partial_{tt}\eta - \delta\partial_{xxtt}\eta + \alpha\,\partial_x^4\eta - \beta\partial_{xx}\eta - \gamma\,\partial_{xxt}\eta = (T_f)_2(\mathbf{u},p,\eta) \text{ in } (0,L), \\ \eta(0,\cdot) = \eta_0,\; \partial_t\eta(0,\cdot) = \eta_1, \end{cases}$$

Boundary conditions:

- $\mathbf{u}(t,\cdot) = \mathbf{0} \quad \text{on } \Gamma_0.$

Coupling conditions:

- $\mathbf{u}(t,x,1+\eta(t,x)) = (0,\partial_t\eta)^T$ on $(0,L)$;
- $\displaystyle\int_0^L T_f \cdot \mathbf{v}(\cdot, 1+\eta(\cdot))\; da = \int_{\Sigma(t)} (-2\nu D(\mathbf{u})\cdot\mathbf{n}_t + p\mathbf{n}_t)\cdot\mathbf{v}\; da_t \qquad \forall \mathbf{v},$

Four cases will be considered

($\mathbf{C_1}$) $\beta > 0,\ \gamma = \delta = \alpha = 0,$

$$\rho_s \partial_{tt}\eta - \beta \partial_{xx}\eta = (T_f)_2,$$

($\mathbf{C_2}$) $\alpha > 0, \delta > 0,\ \beta = \gamma = 0,$

$$\rho_s \partial_{tt}\eta - \delta \partial_{xxtt}\eta + \alpha\, \partial_x^4 \eta = (T_f)_2,$$

($\mathbf{C_3}$) $\alpha > 0, \gamma > 0,\ \beta = \delta = 0,$

$$\rho_s \partial_{tt}\eta + \alpha\, \partial_x^4 \eta - \gamma\, \partial_{xxt}\eta = (T_f)_2,$$

($\mathbf{C_4}$) $\alpha > 0,\ \gamma = \beta = \delta = 0,$

$$\rho_s \partial_{tt}\eta + \alpha\, \partial_x^4 \eta = (T_f)_2,$$

Results

- For the first three cases: existence of strong solutions locally in time, [CG, Hillairet, Lequeurre, 19].
- For the case ($\mathbf{C_3}$): no collision occurs and there exists a unique global in time strong solution, [CG, Hillairet, 16].
- For the four cases: existence of weak solution as long as no contact occurs.
- For the case ($\mathbf{C_4}$): existence of a global weak solution "beyond contact", [Casanova, CG, Hillairet, in progress].

Strong Solutions

Functional spaces of strong solutions

For the fluid:

$$\boldsymbol{u} \in H^1_\sharp(\mathcal{Q}_T)\,, \quad \nabla^2 \boldsymbol{u} \in L^2_\sharp(\mathcal{Q}_T)\,, \quad c \in L^2(0,T)\,, \quad \nabla p \in L^2_\sharp(\mathcal{Q}_T)\,,$$

For the structure:

- In the case $(\mathbf{C_1})$

$$\eta \in H^2(0,T;L^2_\sharp(0,L)) \cap L^\infty(0,T;H^2_\sharp(0,L)) \cap W^{1,\infty}(0,T;H^1_\sharp(0,L)),$$

- In the case $(\mathbf{C_2})$

$$\eta \in H^2(0,T;H^1_\sharp(0,L)) \cap L^\infty(0,T;H^3_\sharp(0,L)) \cap W^{1,\infty}(0,T;H^2_\sharp(0,L)),$$

- In the case $(\mathbf{C_3})$

$$\eta \in H^2(0,T;L^2_\sharp(0,L)) \cap L^2(0,T;H^4_\sharp(0,L)),$$

Steps of the Proof

- Write the fluid equations on the reference configuration Ω_0

$$\rho_f(\det\nabla\chi_\eta)\partial_t\boldsymbol{v}+\rho_f(\boldsymbol{v}\cdot(B_\eta\nabla))\boldsymbol{v}-\mu\mathrm{div}((A_\eta\nabla)\boldsymbol{v})+(B_\eta\nabla)q=0,\quad \text{in } \Omega_0$$

  with $\chi_\eta(x,y)=(x,y(1+\eta_0(x))+\mathcal{R}(\eta-\eta_0)(x,y)$ where $\mathcal{R}$ is a linear operator from $H^{3/2+\varepsilon}_\sharp(0,L)$ into $H^{2+\varepsilon}_\sharp(\Omega_0)$.

- Write the problem as a perturbation with respect to a linear problem,

$$\begin{aligned}\rho_f(\det\nabla\chi_{\eta_0})\partial_t\boldsymbol{v}-\mu\mathrm{div}((A_{\eta_0}\nabla)\boldsymbol{v})+(B_{\eta_0}\nabla)q\\ =\rho_f(\det\nabla\chi_{\eta_0}-\det\nabla\chi_{\eta_0})\partial_t\boldsymbol{v}-\rho_f(\boldsymbol{v}\cdot(B_\eta\nabla))\boldsymbol{v}\\ +\mathrm{div}(((A_{\eta_0}-A_\eta\nabla))\boldsymbol{v})+((B_{\eta_0}-B_\eta)\nabla)q,\quad \text{in } \Omega_0\,.\end{aligned}$$

- Study the linear problem with $\eta_0\in H^2_\sharp(0,L)$.

- Fixed point procedure: the right hand side is small for small time since in particular $\|\eta-\eta_0\|_{L^\infty(0,T;H^{3/2+\varepsilon}_\sharp(0,L))}\leq T^\theta\|\eta-\eta_0\|_{L^\infty(0,T;H^2_\sharp(0,L))\cap W^{1,\infty}(0,T;L^2_\sharp(\Omega_0))}$.

The linear unsteady case

We study the following linear coupled problem (in the case $(\mathbf{C_1})$):

$$\begin{aligned} \rho_f(\det\nabla\chi_{\eta_0})\partial_t\boldsymbol{v} - \mu\mathrm{div}((A_{\eta_0}\nabla)\boldsymbol{v}) + (B_{\eta_0}\nabla)q = f + \mathrm{d}ivh, \ \text{in } \Omega_0, \\ \mathrm{div}(B_{\eta_0}\boldsymbol{v}) = g, \ \text{in } \Omega_0, \\ \rho_s\partial_{tt}\eta - \beta\partial_{xx}\eta = -\nu(A_{\eta_0}\nabla)\boldsymbol{v})e_2\cdot e_2 + qB_{\eta_0}e_2\cdot e_2 + he_2\cdot e_2, \ \text{on } (0,L), \\ \boldsymbol{v}(x,1) = \partial_t\eta\, e_2, \ \text{on } (0,L). \end{aligned}$$

Remark: The right hand side g satisfies the compatibility condition $\int_{\Omega_0} g = \int_0^L \partial_t\eta = 0$.

The linear unsteady case

Key steps

- We keep the system as a coupled system,
- First step: lifting the non homogeneous divergence g keeping the kinematic boundary condition $\boldsymbol{v}(x,1) = \partial_t \eta\, e_2$. It is made possible because of $\int_{\Omega_0} g = 0$,
- $(\boldsymbol{v}, \partial_t \eta)$ as test functions $\Longrightarrow$ energy estimates,
- $(\partial_t \boldsymbol{v}, \partial_{tt}\eta)$ as test functions $\Longrightarrow$ estimates of the accelerations in L^2 provided that we can control $\|\partial_{xt}\eta\|^2_{L^2(0,T;L^2_\sharp(0,L))}$,
- $(-\partial_{xx}\boldsymbol{v}, -\partial_{xxt}\eta)$ as test functions $\Longrightarrow$ estimate on $\|\nabla \partial_x \boldsymbol{v}\|^2_{L^2(0,T;H^1_\sharp(\Omega_0))}$ and on $\|\partial_{xt}\eta\|^2_{L^\infty(0,T;L^2_\sharp(0,L))}$. But we have to estimate pressure terms as
$$\int_{\Omega_0} \partial_x q\, \partial_x B_{\eta_0} : \mathrm{div}\, \boldsymbol{v}$$

$\Longrightarrow$ Use of the elliptic estimate for the Stokes problem that requires a control of $\partial_t \boldsymbol{v}$ in $L^2_\sharp(\Omega_0)$ and $\partial_t \eta$ in $H^{3/2}_\sharp(0,L)$.

Elliptic Result

Elliptic Result

Let $b \in H^2_\sharp(0,L)$, $\min_{x\in[0,L]}(1+b(x)) > 0$, consider the Stokes problem in

$$\Omega_b = \{(x,y) \in \mathbb{R}^2, x \in (0,L), y \in (0,1+b(x))\}$$

$$\begin{aligned} -\nu\Delta \boldsymbol{v} + \nabla q &= f, \quad \text{in } \Omega_b, \\ \operatorname{div} \boldsymbol{v} &= g, \quad \text{in } \Omega_b, \\ \boldsymbol{v}(x, 1+b(x)) &= (0, \dot\eta(x)), x \in (0,L). \end{aligned}$$

The study of the elliptic regularity of $(\boldsymbol{v}, q)$ is equivalent to the one for $(\boldsymbol{z}, r)$ defined by

$$\boldsymbol{z}(x,y) = \boldsymbol{v}(x, y(1+b(x)), \quad r(x,y) = q(x, y(1+b(x))$$

solution of

$$\begin{aligned} -\nu\operatorname{div}(A_b\nabla \boldsymbol{z}) + (B_b\nabla)r &= \bar f, \quad \text{in } \Omega_0, \\ \operatorname{div}(B_b^t \boldsymbol{z}) &= \bar g, \quad \text{in } \Omega_0, \\ \boldsymbol{z}(x,1) &= (0,\dot\eta(x)), \quad x \in (0,L), \end{aligned}$$

with $B_b = \operatorname{cof} \nabla\chi_b$, $A_b = \dfrac{B_b B_b^t}{1+b}$, and $\chi_b(x,y) = (x, y(1+b(x)))$.

Elliptic Result

Let $b \in H^2_\sharp(0,L)$, $\min_{x\in[0,L]}(1+b(x)) > 0$, consider the Stokes problem in

$$\Omega_b = \{(x,y) \in \mathbb{R}^2, x \in (0,L), y \in (0, 1+b(x))\}$$

$$\begin{aligned} -\nu\Delta \boldsymbol{v} + \nabla q &= f, \quad \text{in } \Omega_b, \\ \operatorname{div} \boldsymbol{v} &= g, \quad \text{in } \Omega_b, \\ \boldsymbol{v}(x, 1+b(x)) &= (0, \dot{\eta}(x)), x \in (0,L). \end{aligned}$$

The study of the elliptic regularity of $(\boldsymbol{v}, q)$ is equivalent to the one for $(\boldsymbol{z}, r)$ defined by

$$\boldsymbol{z}(x,y) = \boldsymbol{v}(x, y(1+b(x)), \quad r(x,y) = q(x, y(1+b(x))$$

solution of

$$\begin{aligned} -\nu \operatorname{div}(A_b \nabla \boldsymbol{z}) + (B_b\nabla)r &= \bar{f}, \quad \text{in } \Omega_0, \\ \operatorname{div}(B_b^t \boldsymbol{z}) &= \bar{g}, \quad \text{in } \Omega_0, \\ \boldsymbol{z}(x,1) &= (0, \dot{\eta}(x)), \quad x \in (0,L), \end{aligned}$$

with $B_b = \operatorname{cof} \nabla\chi_b$, $A_b = \dfrac{B_b B_b^t}{1+b}$, and $\chi_b(x,y) = (x, y(1+b(x)))$.

Elliptic Result: Idea of the proof

For $b \in H^2_\sharp(0,L)$, $\min_{x\in[0,L]}(1+b(x)) > 0$, if $f \in L^2_\sharp(\Omega_0)$, $\bar{g} \in H^1_\sharp(\Omega_0)$, $\dot{\eta} \in H^{3/2}_\sharp(0,L)$ and if the compatibility condition $\displaystyle\int_{\Omega_0} \bar{g} = \int_0^L \dot{\eta}$ is satisfied then $z \in H^2_\sharp(\Omega_0)$ and $r \in H^1_\sharp(\Omega_0)$.

Steps of the Proof:

- Existence a unique of weak solution and estimates with respect to the data with $\bar{f} \in H^{-1}_\sharp(\Omega_0)$, $\bar{g} \in L^2_\sharp(\Omega_0)$ and $\dot{\eta} \in H^{1/2}_\sharp(0,L)$.
  Key arguments:
  - A_b is coercive, $A_b \geq \lambda I$,
  - B_b is invertible.

Elliptic Result: Idea of the proof

- Regularity estimates.

  Key arguments:

  – Differentiate the equation with respect to x. $(\partial_x \boldsymbol{z}, \partial_x r)$ is solution of

$$\begin{aligned}
-\nu \mathrm{div}(A_b \nabla \partial_x \boldsymbol{z}) + (B_b \nabla)\partial_x r &= \partial_x \bar{f} + \nu \mathrm{div}(\partial_x A_b \nabla \boldsymbol{z}) - (\partial_x B_b \nabla) r, \quad \text{in } \Omega_0, \\
\mathrm{div}(B_b^t \partial_x \boldsymbol{z}) &= \partial_x \bar{g} - \mathrm{div}(\partial_x B_b^t \boldsymbol{z}), \quad \text{in } \Omega_0, \\
\partial_x \boldsymbol{z}(x,1) &= (0, \partial_x \dot{\eta}(x)), \quad x \in (0,L),
\end{aligned}$$

  – Use the previous step together with
    * $\partial_x A_b \in L^\infty(0,1; H^1_\sharp(0,L))$,
    * $\mathrm{div}(\partial_x A_b \nabla \boldsymbol{z}) \in H^{-1}_\sharp(\Omega_0)$, $\mathrm{div}(\partial_x B_b^t \boldsymbol{z}) = \partial_x B_b^t : \nabla \boldsymbol{z} \in L^2_\sharp(\Omega_0)$,
    * $\|\mathrm{div}\ ((\partial_x A_b \nabla)\boldsymbol{z})\|_{H^{-1}_\sharp(\Omega_0)} \le \|(\partial_x A_b \nabla)\boldsymbol{z}\|_{L^2_\sharp(\Omega_0)} \le K(b) \|\boldsymbol{z}\|^{1/2}_{H^1_\sharp(\Omega_0)} \|\partial_x \boldsymbol{z}\|^{1/2}_{H^1_\sharp(\Omega_0)}$.

Key Ingredients

- Obtain elliptic results for the Stokes problem;
- Control the geometrical terms;
- Fill the gap created by the parabolic-hyperbolic coupling $\Longrightarrow$ use the damping coming from the fluid;
- Avoid the added mass effect $\Longrightarrow$ keep the problem as a coupled system or decouple it in a proper way;

Other results

- [H. Beirão Da Veiga, 04] Existence of strong solutions for a 2D fluid, locally in time, for small data

- [J. Lequeurre, 10, 12] Existence locally in time of strong solutions for 2D or 3D fluid coupled with a plate or a membrane with additional viscosity on the structure

Idea: Decouple and use the regularity of the solution of each sub–problem
$\Longrightarrow$ Added Mass effect

Sketch of the proof of J. Lequeurre

Write the fluid equations in the reference configuration:

Fluid:

$$\begin{cases} \rho^f \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = F_\eta(\mathbf{u}, p) & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = G_\eta(u) & \text{in } \Omega, \\ \mathbf{u}(0, \cdot) = \mathbf{u}_0 & \text{in } \Omega, \end{cases}$$

Structure:

$$\begin{cases} \rho^s \partial_{tt}\eta + \alpha \partial_x^4 \eta - \beta \partial_{xx}\eta - \gamma \partial_x \partial_t \eta \\ = p + (T_\eta(u,p))_2 \text{ in } \omega = (0, L), \\ \eta = \dfrac{\partial \eta}{\partial n} = 0 \text{ on } \partial\omega, \\ \eta(0, \cdot) = \eta_0, \ \partial_t \eta(0, \cdot) = \eta_1, \end{cases}$$

Coupling conditions: Equality of the velocities:

$$\mathbf{u}(t, x, 1) = (0, \partial_t \eta(t, x))^T, x \in \omega.$$

Steps

- Study of the linear problem by decoupling the fluid and the structure. But first:
  - Lifting of the non homogeneous divergence problem. Works well since $\int_\Omega G_\eta(u) = 0$
  - Split the fluid pressure and velocity in two parts, one of which will be treated implicitly.
- Prove that the non linear terms are small, for small enough time.

Fluid:

Define $\mathbf{u}_s$ as $\mathbf{u}_s = \nabla\theta$

$$\left\{\begin{array}{l} \Delta\theta = 0 \\ \dfrac{\partial\theta}{\partial n} = \partial_t\eta \end{array}\right.$$

Define p_s as

$$\left\{\begin{array}{l} \Delta p_s = \text{div } \mathbf{f} \\ \dfrac{\partial p_s}{\partial n} = -\rho_f\partial_{tt}\eta + f.n \end{array}\right.$$

Define $(\mathbf{u}_e, p_e)$ as

$$\left\{\begin{array}{l} \rho^f\partial_t\mathbf{u}_e - \nu\Delta\mathbf{u}_e + \nabla p_e = \Pi(\mathbf{f}), \\ \\ \text{div}\,\mathbf{u}_e = 0, \\ \\ \mathbf{u}_e\cdot\mathbf{n} = 0 \\ \mathbf{u}_e\cdot\tau = -\mathbf{u}_s\cdot\tau, \end{array}\right.$$

Structure:

$$\left\{\begin{array}{l} (\rho^s + \rho_f\mathcal{M})\partial_{tt}\eta + \alpha\partial_x^4\eta - \beta\partial_{xx}\eta - \gamma\partial_x\partial_t\eta = p_e + H \text{ in } \omega, \\ \\ \eta = \dfrac{\partial\eta}{\partial n} = 0 \text{ on } \partial\omega, \\ \eta(0,\cdot) = \eta_0,\ \partial_t\eta(0,\cdot) = \eta_1, \end{array}\right.$$

Ref: [J.-P. Raymond]

Global existence of strong solutions

Global existence of strong solution

Questions:

- Either we are able to give sense to the solution after "collision"?
- Or we can prove that no collision occur?

Collision in the rigid body case [Starovoitov], [San Martin et al], [Hillairet], [Hillairet, Takahashi], [Gérard-Varet, Hillairet]...

- 2D: no contact for weak solutions and smooth rigid body [Hillairet]
- 3D: no contact for weak solutions and smooth rigid body, and contact between the ball and the boundary of the cavity implies blow up of strong solutions [Hillairet, Takahashi]
- 2D: body with $C^{1,\alpha}, \alpha < 1/2$ boundary, contact is possible [Gérard-Varet, Hillairet]

No-collision and Global Existence Result

We consider the case **(C_3)**, namely a beam in flexion with additional viscosity. We know that the exists of a unique strong solution locally in time, s. t.

$$\boldsymbol{u} \in L^2(0,T;H^2_\sharp(\Omega_\eta(t))) \cap H^1(0,T;L^2_\sharp(\Omega_\eta(t)))$$

$$p \in L^2(0,T;H^1_\sharp(\Omega_\eta(t)))$$

$$\eta \in H^2(0,T;L^2_\sharp(0,L)) \cap L^2(0,T;H^4_\sharp(0,L))$$

The solution blows up in finite time if and only if the quantity

$$\mathcal{C}(t) := \sup_{x\in[0,1]} \frac{1}{h(x,t)} + \int_0^L \left(\alpha|\partial_{xxx}h(x,t)|^2 + \gamma|\partial_{tx}h(x,t)|^2\right) \mathrm{d}x \\ + \int_0^L \int_0^{h(x,t)} \mu|\nabla \boldsymbol{u}(x,y,t)|^2 \mathrm{d}x\mathrm{d}y$$

with $h = 1+\eta$, blows up in finite time.

No-collision Result

$$h \in L^\infty(0,T;H^2_\sharp(0,L))$$

$$\int_0^L \frac{1}{h} < +\infty$$

$\Longrightarrow$ prevents contact;

Idea: write $h(x) = h(x_0) + \int_{x_0}^{x} (s-x)\partial_{xx} h(s) \mathrm{ds}$.

To obtain this additional estimate we need $\alpha > 0$ and $\gamma > 0$.

$\Longrightarrow$ the elliptic regularity result holds true

$\Longrightarrow$ we can prove the existence of strong solution on any time interval $[0,T]$.

Simplified model

$$\begin{aligned} \partial_{tt}\rho - \beta\partial_{xx}\rho + \alpha\,\partial_{xxxx}\rho - \gamma\,\partial_{xxt}\rho &= q, & (1)\\ \partial_t\rho &= \partial_x[\rho^3\partial_x q]\,, & (2) \end{aligned}$$

First, multiplying (1) by $\partial_t\rho$ and combining with (2) multiplied by q yields:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^1 \left(|\partial_t\rho|^2 + \beta|\partial_x\rho|^2 + \alpha|\partial_{xx}\rho|^2\right)\right] + \int_0^1 \left(\gamma|\partial_{tx}\rho|^2 + \rho^3|\partial_x q|^2\right) = 0\,.$$

$\Longrightarrow$ Energy estimates

Then, we multiply (1) by $-\partial_{xx}\rho$. Integration by parts yields, applying (2) to compute $\partial_x[\rho^3\partial_x q]$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^1 \left(\frac{\gamma}{2}|\partial_{xx}\rho|^2 - \partial_t\rho\,\partial_{xx}\rho\right)\right] + \int_0^1 \left(\beta|\partial_{xx}\rho|^2 + \alpha|\partial_{xxx}\rho|^2 - |\partial_{tx}\rho|^2\right)$$
$$= \int_0^1 q\partial_{xx}\rho = -\frac{1}{2}\int_0^1 \rho^3\partial_x q\partial_x\left[\frac{1}{\rho^2}\right] = \frac{1}{2}\int_0^1 \frac{\partial_t\rho}{\rho^2}\,.$$

Finally, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^1 \frac{1}{2}\left(\gamma|\partial_{xx}\rho|^2 + \frac{1}{\rho}\right) - \int_0^1 \partial_t\rho\,\partial_{xx}\rho\right] + \int_0^1 \left(\beta|\partial_{xx}\rho|^2 + \alpha|\partial_{xxx}\rho|^2\right) = \int_0^1 |\partial_{tx}\rho|^2\,.$$

Simplified problem

The uniform lower bound of ρ uniformly in time and the dissipation estimate leads to

$$\sup_{t\in(0,T)}\left(\|\rho^{-1}(\cdot,t)\|_{L^\infty_\sharp(0,1)}\right)+\int_0^T\|q\|^2_{H^1_\sharp(0,1)}\leq C_0\,.$$

We can now multiply the structure equation by $-\partial_{txx}\rho$. This yields after integration by parts:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^1|\partial_{tx}\rho|^2+\alpha|\partial_{xxx}\rho|^2+\beta|\partial_{xx}\rho|^2\right]+\gamma\int_0^1|\partial_{txx}\rho|^2=\int_0^1 q\partial_{txx}\rho.$$

Because of the H^1-bound on q, we obtain a global regularity estimate:

$$\sup_{t\in(0,T)}\left(\|\rho\|^2_{H^3_\sharp(0,1)}+\|\partial_t\rho\|^2_{H^1_\sharp(0,1)}\right)+\int_0^T\|\partial_t\rho\|^2_{H^2_\sharp(0,1)}\leq C_0\,.$$

Idea of the proof: each estimate is done on the system written is the deformed configuration

- Energy estimate – Test functions: $(\boldsymbol{u}, \partial_t \eta)$.
  $\Longrightarrow \boldsymbol{u} \in L^2(0,T; H^1_\sharp(\Omega_\eta(t))) \cap L^\infty(0,T; L^2_\sharp(\Omega_\eta(t)))$,
  $\eta \in L^\infty(0,T; H^2_\sharp(0,L)) \cap W^{1,\infty}(0,T; L^2_\sharp(0,L)) \cap H^1(0,T; H^1_\sharp(0,L))$.
- Distance estimate - Choice of appropriate test functions:

$$\boldsymbol{w} = \nabla^\perp \underbrace{(\partial_x \eta \; \chi(\frac{y}{1+\eta}))}_{\psi}, \quad \text{where } \chi(t) = t^2(3-2t)$$

and

$$q = \partial_{xy}\psi - \int_0^x \partial_{y^3}\psi$$

$\Longrightarrow \boldsymbol{w}(t,x,1+\eta(x)) = \partial_{xx}\eta(t,x)$.

Thus:

$$\int_0^L \text{Structure equation } \times \partial_{xx}\eta = \underbrace{\int_0^L (T_f e_2)\cdot e_2 \, \partial_{xx}\eta}_{\text{to be estimated}}$$

$$= \int_{\Omega_\eta(t)} \partial_t \boldsymbol{u} \cdot \boldsymbol{w} + \nu \underbrace{\int_{\Omega_\eta(t)} \nabla \boldsymbol{u} : \nabla \boldsymbol{w}} + \int_{\Omega_\eta(t)} (\boldsymbol{u}\nabla)\boldsymbol{u}\boldsymbol{w}$$

$$\boxed{\int_{\partial\Omega_\eta(t)} T_f(\boldsymbol{w}, q)\mathbf{n}\cdot\boldsymbol{u} - \int_{\Omega_\eta(t)} (\nu\Delta\boldsymbol{w} - \nabla q)\boldsymbol{u}}$$

There exists a constant C_0 depending only on initial data for which:

$$\sup_{t\in(0,T)} \Big(\gamma\|h(t,\cdot)\|^2_{H^2_\sharp(0,L)} + \|h^{-1}(t,\cdot)\|_{L^1_\sharp(0,L)}\|\Big)$$
$$+\alpha\int_0^T \|h(t,\cdot)\|^2_{H^3_\sharp(0,L)}\mathrm{d}t \le C_0(1+T)\,.$$

Remarks:

- Gives an estimate of $\partial_{x^3} h$ in $L^2(0,T\times(0,L))$ (because $\alpha>0$),
- Need to control $\partial_{x^2} h$ in $L^\infty(0,L)$ (thus $\alpha>0$),
- Need to have dissipation in the beam equation (thus $\gamma>0$).

Regularity Estimates

- Natural fluid-structure test functions $(\partial_t \boldsymbol{u}, \partial_{tt}\eta)$.

  But these are not appropriate test functions (unlike for the linear problem)

  $\Longrightarrow$ take into account the domain motion
  $\Longrightarrow$ need of some additional regularity of the structure displacement

- Additional estimate for the structure: Structure test function: $-\partial_{txx}\eta$

At this step, need to have an elliptic estimate.

Choice of the tests functions:

$$\mathbf{v} := \partial_t \mathbf{u} + \hat{\Lambda} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \hat{\Lambda}$$

with

- $\hat{\Lambda}$ is divergence free,
- $\nabla \hat{\Lambda} \in L^2_\sharp(\mathcal{Q}_T)$ and $\nabla^2 \hat{\Lambda} \in L^2_\sharp(\mathcal{Q}_T)$ with

$$\begin{aligned}
\|\hat{\Lambda}(\cdot,t)\|_{L^2_\sharp(\Omega_\eta)} &\leq C_0(t)\|\partial_t \eta(\cdot,t)\|_{L^2_\sharp(0,L)}\,, && (1)\\
\|\nabla\hat{\Lambda}(\cdot,t)\|_{L^2_\sharp(\Omega_\eta)} &\leq C_0(t)\|\partial_t \eta(\cdot,t)\|_{H^1_\sharp(0,L)}\,, && (2)\\
\|\nabla^2\hat{\Lambda}(\cdot,t)\|_{L^2_\sharp(\Omega_\eta)} &\leq C_0(t)\|\partial_t \eta(\cdot,t)\|_{H^2_\sharp(0,L)}\,, && (3)
\end{aligned}$$

  for a.e. $t \in (0,T)$,
- $\hat{\Lambda} = \mathbf{u}$ and $\partial_2 \hat{\Lambda} = 0$ on $y = h(x,t)$,
- $\hat{\Lambda} = 0$ on $y = 0$.

- To estimate the convention terms $\Longrightarrow$ need of an elliptic estimate;
- Associated structure test function: $\partial_{tt}\eta$

  $\Longrightarrow$ Need to control the term $\displaystyle\int_0^L |\partial_x^2 \partial_t \eta|^2$

  $\Longrightarrow$ Need to obtain an additional estimate for the structure part.
  Test function: $-\partial_{xx}\partial_t\eta$

  $\Longrightarrow$ need to control the applied load

  $\Longrightarrow$ elliptic estimate.

Weak solutions

Weak solutions

- Existence of weak solution as long as the structure does not touch the bottom of the fluid cavity: the case of the wave equation with and without damping ($\beta > 0, \gamma \geq 0$);

- Existence of weak solution "beyong contact": the case of the beam equation without damping ($\alpha > 0$).

Definition of the functional spaces

Note that the following continuous injection holds :

$$W^{1,\infty}(0,T;L^2(0,L)) \cap L^\infty(0,T;H^1(0,L)) \hookrightarrow C^{0,1-\theta}([0,T];H^\theta(0,L)),$$

for all $0 < \theta < 1$. In particular,

$$W^{1,\infty}(0,T;L^2(0,L)) \cap L^\infty(0,T;H^1(0,L)) \hookrightarrow C^{0,\mu}([0,T];C^{0,1/2-\theta}(0,L)),$$

for all $0 < \theta < 1/2$.
We define :

$$L^2(0,T;H^1(\Omega_\eta(t))) = \left\{ v \in L^2(\widehat{\Omega}_\eta),\ \nabla v \in L^2(\widehat{\Omega}_\eta) \right\},$$

$$L^2(0,T;H^1_0(\Omega_\eta(t))) = \overline{\mathcal{D}(\widehat{\Omega}_\eta)}^{L^2(0,T;H^1(\Omega_\eta(t)))},$$

$$\mathcal{V}_\delta = \left\{ \boldsymbol{v} \in C^1\left(\overline{\widehat{\Omega}_\eta}\right),\ \mathrm{div}\boldsymbol{v} = 0,\ \boldsymbol{v} = \boldsymbol{0} \text{ on } (0,T) \times \Gamma_0 \right\},$$

$$V_\eta = \overline{\mathcal{V}_\eta}^{L^2(0,T;H^1(\Omega_\eta(t)))},$$

with $\widehat{\Omega}_\eta = \cup_t \{t\} \times \Omega_\eta(t)$.

Trace on the interface

The trace of the fluid velocity $\mathbf{u}$ on the moving interface is not defined since the interface is not lipschitz.
But $\mathbf{u}(t, x, 1 + \eta(t, x))$ is well defined in $L^2(0, L)$ for a. e. t.

Indeed, let $v \in H^1(\Omega_\eta(t))$, $v = 0$ on Γ_0.

$$
\begin{aligned}
\|v(x, 1 + \eta(t, x))\|^2_{L^2(0,L)} &= \int_0^L (v(x, 1 + \eta(t, x))^2 dx \\
&= \int_0^L \left(\int_0^{1+\eta(t,x)} \partial_z v(x, u) du \right)^2 dx
\end{aligned}
$$

Lifting of the structure test functions

Let $b \in H^1_0(0,L)$. We define

$$\tilde{\mathbf{v}} = \left| \begin{array}{l} (0,b)^T \text{ in } \Omega_\eta(t) \setminus \mathcal{C}_\alpha \\ \mathcal{R}(0,b)^T \text{ in } \mathcal{C}_\alpha \end{array} \right.$$

then $\tilde{\mathbf{v}}$ belongs to $H^1(\Omega_\eta(t))$ and $\mathrm{div}(\tilde{\mathbf{v}}) = 0$.

Extention of fluid test functions

Let $\mathbf{v}$ be such that $\mathrm{div}(\mathbf{v}) = 0$ and $\mathbf{v}(x, 1+\eta(x)) = (0, b(x))^T$ on $(0,L)$. We define

$$\overline{\mathbf{v}} = \left| \begin{array}{l} \mathbf{v} \text{ in } \Omega_\eta(t) \\ (0,b)^T \text{ in } B \setminus \Omega_\eta(t) \end{array} \right.$$

Then $\overline{\mathbf{v}}$ belongs to $H^1(B)$ and $\mathrm{div}(\overline{\mathbf{v}}) = 0$.

Existence theorem

We assume that

$$\mathbf{u}_0 \in L^2(\Omega_{\eta_0}), \eta_0 \in H^1_0(0,L), \eta_1 \in L^2(\omega)$$

and

$$\begin{aligned}
&\min_{\overline{\omega}}(1+\eta_0) > 0, \operatorname{div} \mathbf{u}_0 = 0 \text{ in } \Omega_{\eta_0}, \\
&\mathbf{u}_0 \cdot \mathbf{n} = 0 \text{ in } \Gamma_0, \gamma^n_{\eta_0}(\mathbf{u}_0) = (0,\eta_1)^T \cdot \mathbf{n}_0 \text{ on } \omega, \\
&\int_0^L \eta_1(x) = 0.
\end{aligned}$$

Result : Let $\mathbf{f} \in L^2_{loc}((0,+\infty)\times\mathbb{R}^3)$, $g \in L^2_{loc}((0,+\infty)\times\omega)$, there exists at least one weak solution as long as the structure does not touch the bottom of the fluid cavity.

Weak formulation

- $\mathbf{u}(t,x,1+\eta(t,x)) = (0, \partial_t\eta(t,x))^T$, $(t,x) \in [0,T] \times (0,L)$,
- $\mathrm{div}(\mathbf{u}) = 0$ in $\widehat{\Omega}_\eta = \bigcup_{t\in(0,T)} \{t\} \times \Omega_\eta(t)$.
- for all $(\boldsymbol{\phi}, b)$ such that
  - $\boldsymbol{\phi}(t,x,1+\eta(t,x)) = (0,0,b(t,x))^T$, $(t,x) \in [0,T] \times (0,L)$,
  - $\mathrm{div}(\boldsymbol{\phi}) = 0$ in $\widehat{\Omega}_\eta$.

we have, for a. e. t

$$\int_{\Omega_\eta(t)} \mathbf{u}(t)\cdot\boldsymbol{\phi}(t) - \int_0^t \int_{\Omega_\eta(s)} \mathbf{u}\cdot\partial_t\boldsymbol{\phi} + \nu \int_0^t \int_{\Omega_\eta(s)} \nabla\mathbf{u} : \nabla\boldsymbol{\phi} + \int_0^t \int_{\Omega_\eta(s)} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\phi}$$
$$- \int_0^t \int_0^L (\partial_t\eta)^2 b + \int_0^L \partial_t\eta(t)\, b(t) - \int_0^t \int_0^L \partial_t\eta\, \partial_t b + \gamma \int_0^t \int_0^L \partial_{tx}\eta\, \partial_x b +$$
$$+ \beta \int_0^t \int_0^L \partial_x\eta\, \partial_x b$$
$$= \int_0^t \int_{\Omega_\eta(t)} \mathbf{f}\cdot\boldsymbol{\phi} + \int_0^t \int_0^L g\ b + \int_{\Omega_{\eta 0}} \mathbf{u}_0\boldsymbol{\phi}(0) + \int_0^L \eta_1 b(0)$$

Weak formulation of the approximated problems

$$\int_0^t \int_{\Omega_{\eta_\varepsilon^*}(s)} \partial_t \mathbf{u}_\varepsilon \cdot \boldsymbol{\phi}_\varepsilon + \nu \int_0^t \int_{\Omega_{\eta_\varepsilon^*}(s)} \nabla \mathbf{u}_\varepsilon : \nabla \boldsymbol{\phi}_\varepsilon$$

$$+ \frac{1}{2} \int_0^t \int_{\Omega_{\eta_\varepsilon^*}(s)} (\mathbf{u}_\varepsilon^* \cdot \nabla) \mathbf{u}_\varepsilon \cdot \boldsymbol{\phi}_\varepsilon - \frac{1}{2} \int_0^t \int_{\Omega_{\eta_\varepsilon^*}(s)} (\mathbf{u}_\varepsilon^* \cdot \nabla) \boldsymbol{\phi}_\varepsilon \cdot \mathbf{u}_\varepsilon$$

$$+ \frac{1}{2} \int_0^t \int_0^L \partial_t \eta_\varepsilon \partial_t \eta_\varepsilon^* b + \int_0^t \int_0^L \partial_{tt} \eta_\varepsilon \, b + \gamma \int_0^t \int_0^L \partial_{tx} \eta_\varepsilon \, \partial_x b$$

$$+ \beta \int_0^t \int_0^L \partial_x \eta_\varepsilon \, \partial_x b = \int_0^t \int_{\Omega_{\eta_\varepsilon^*}(s)} \mathbf{f} \cdot \boldsymbol{\phi}_\varepsilon + \int_0^t \int_0^L g \; b,$$

$$\forall (\boldsymbol{\phi}_\varepsilon, b) \text{ such that } \operatorname{div}(\boldsymbol{\phi}_\varepsilon) = 0$$

$$\boldsymbol{\phi}_\varepsilon(t, x, 1 + \eta_\varepsilon^*(t,x)) = (0, b(t,x))^T \text{ on } (0, L).$$

Weak formulation of the linearized-approximated problems

$$\int_0^t \int_{\Omega_{\delta_\varepsilon^*}(s)} \partial_t \mathbf{u}_\varepsilon \cdot \boldsymbol{\phi}_\varepsilon + \nu \int_0^t \int_{\Omega_{\delta_\varepsilon^*}(s)} \nabla \mathbf{u}_\varepsilon : \nabla \boldsymbol{\phi}_\varepsilon$$

$$+ \frac{1}{2} \int_0^t \int_{\Omega_{\delta_\varepsilon^*}(s)} (\mathbf{v}_\varepsilon^* \cdot \nabla) \mathbf{u}_\varepsilon \cdot \boldsymbol{\phi}_\varepsilon - \frac{1}{2} \int_0^t \int_{\Omega_{\delta_\varepsilon^*}(s)} (\mathbf{v}_\varepsilon^* \cdot \nabla) \boldsymbol{\phi}_\varepsilon \cdot \mathbf{u}_\varepsilon$$

$$+ \frac{1}{2} \int_0^t \int_0^L \partial_t \eta_\varepsilon \partial_t \delta_\varepsilon^* b + \int_0^t \int_0^L \partial_{tt} \eta_\varepsilon \, b + \gamma \int_0^t \int_0^L \partial_{tx} \eta_\varepsilon \, \partial_x b$$

$$+ \beta \int_0^t \int_0^L \partial_x \eta_\varepsilon \, \partial_x b = \int_0^t \int_{\Omega_{\delta_\varepsilon^*}(s)} \mathbf{f} \cdot \boldsymbol{\phi}_\varepsilon + \int_0^t \int_0^L g \; b,$$

$$\forall (\boldsymbol{\phi}_\varepsilon, b) \text{ such that } \operatorname{div}(\boldsymbol{\phi}_\varepsilon) = 0$$

$$\boldsymbol{\phi}_\varepsilon(t, x, 1 + \delta_\varepsilon^*(t,x)) = (0, b(t,x))^T \text{ on } (0, L).$$

Sketch of the proof

- Galerkin method
  Fluid test functions: $\{\mathbf{v} \in H^1_0, \mathrm{div}(\mathbf{v}) = 0\}$ + a lifting of structure test functions $\longrightarrow$ existence for the linearized-approximated problem

- Fixed point procedure
  $(\mathbf{v}, \delta) \mapsto (\mathbf{v}^*_\varepsilon, \delta^*_\varepsilon) \to (\mathbf{u}_\varepsilon, \eta_\varepsilon) \longrightarrow$ existence for the approximated problem.

- $\varepsilon \longrightarrow 0$, $\mathbf{u}_\varepsilon \to \mathbf{u}$, $\eta_\varepsilon \to \eta \longrightarrow$ existence for the initial problem.

In order to pass to the limit in the equation one needs a compactness result

Lemma. *Let $T > 0$ be such that $\min_{[0,T]\times[0,L]} (1 + \eta_\varepsilon) \geq \alpha > 0$. For every $h > 0$, $h \leq h_0$*

$$\int_0^T \int_B \rho_\varepsilon |\overline{\mathbf{u}}_\varepsilon - \overline{\mathbf{u}}_\varepsilon^-|^2 + \int_0^T \int_0^L (\partial_t \eta_\varepsilon - \partial_t \eta_\varepsilon^-)^2 \leq C\sqrt{h},$$

and

$$\int_0^T \int_B |\rho_\varepsilon \overline{\mathbf{u}}_\varepsilon - \rho_\varepsilon^- \overline{\mathbf{u}}_\varepsilon^-|^2 \leq C\sqrt{h},$$

where ρ_ε denotes the characteristic function of $\Omega_{\eta^*_\varepsilon}(t)$ and $v(t) = v(t-h)$.

Linearized-appromimated problem

We write the full coupled weak formulation in the reference configuration thanks to the transformation:

$$\phi_\varepsilon(t,x,y) = (x, y(1+\delta_\varepsilon^*(t,x)))$$

$$\Longrightarrow$$

$$-\boldsymbol{\Delta u}_\varepsilon \longrightarrow -\operatorname{div}\,(A_\varepsilon \nabla \boldsymbol{u}_\varepsilon \circ \phi_\varepsilon),$$

$$\partial_t \boldsymbol{u}_\varepsilon \longrightarrow \partial_t \boldsymbol{u}_\varepsilon \circ \phi_\varepsilon J_\varepsilon - \boldsymbol{w}_\varepsilon (B_\varepsilon^T \nabla) \boldsymbol{u}_\varepsilon \circ \phi_\varepsilon ...$$

Construction of a Galerkin Basis

- For the structure: $(\xi_i)_i$ Galerkin basis associated to the space

$$\{b \in H_0^1(0,L), s.t. \int_0^L b = 0\}.$$

- For the fluid:
  $(\psi_i^0)_i$ a Galerkin basis of $\{\boldsymbol{v} \in H_0^1(\Omega_f), s.t.\operatorname{div}\,(B_\varepsilon^T \boldsymbol{v}) = 0\}$

  $+\ \psi_\varepsilon^1$ lifting of the structure test functions such that div $(B_\varepsilon \psi_\varepsilon^1) = 0$.

  Remark: The discrete system reduces to a system on the fluid degree of freedom.

Approximated problem

Choice of test functions for the compactness result to pass to the limit as ε goes to zero:

$$\boldsymbol{\phi}_\varepsilon = \int_{t-h}^{t} (\overline{\mathbf{u}}_\varepsilon)_\sigma(s)ds, \quad b = \int_{t-h}^{t} \partial_t \eta_\varepsilon(s)ds.$$

where

$$\mathbf{v}_\sigma(x,y,z) = (\sigma \mathbf{v}_1(x,\sigma y), \mathbf{v}_2(x,\sigma y))$$

with

$$\sigma \geq 1 + C\sqrt{h}.$$

Remark: If $\mathbf{v}$ is divergence free, $\mathbf{v}_\sigma$ is also divergence free.

$$\varepsilon \longrightarrow 0$$

Main points:

- Pass to the limit in the domain sequence
- Choose test functions independent of ε

$\boxed{\textbf{And } \gamma \longrightarrow 0?}$

Result: *When γ goes to zero $(\mathbf{u}_\gamma,\, \eta_\gamma)$ converge towards $(\mathbf{u}, \eta)$, weak solution of the coupled problem: Navier-Stokes/ Euler Bernoulli. We obtain the existence of at least one weak solution as long as the plate does not touch the bottom of the fluid cavity.*

Idea: Split the low and high frequencies of the structure velocity

$\Longrightarrow$ Low frequencies controlled as previously

$\Longrightarrow$ High frequencies controlled thanks to the dissipation coming from the fluid and since

$$\boldsymbol{u}(t, x, 1 + \eta(t, x)) = (0, \partial_t \eta(t, x))^T \in L^2(0, T; H^s(\omega)), \text{for some } s > 0$$

Sketch of the proof

- $(T_\gamma)_\gamma$ bounded from below.
- Compactness

Lemma. *Let* $T>0$ *be such that* $\min_{[0,T]\times[0,L]}(1+\eta_\gamma)\geq\alpha>0$. *Then when* h *goes to* 0, *we have*

$$\int_0^T\int_B \rho_\gamma|\overline{\mathbf{u}}_\gamma-\overline{\mathbf{u}}_\gamma^-|^2+\int_0^T\int_\omega(\partial_t\eta_\gamma-\partial_t\eta_\gamma^-)^2\longrightarrow 0,$$

and

$$\int_0^T\int_B|\rho_\gamma\overline{\mathbf{u}}_\gamma-\rho_\gamma^-\overline{\mathbf{u}}_\gamma^-|^2\longrightarrow 0,$$

Choice of the tests functions

$$\boldsymbol{\phi}_\gamma=\int_{t-h}^t(\overline{(\mathbf{u}_\gamma-\mathcal{R}((I-\Pi_{N_0})(\partial_t(\eta_\gamma(s))))}_\sigma(s)ds,\quad b=\int_{t-h}^t\Pi_{N_0}(\partial_t(\eta_\gamma(s)))ds.$$

Weak solutions with contact for a beam without viscosity

Consider $\alpha > 0$ and $\gamma > 0$ for which we have existence of global strong solution.

- Question: Can we pass to the limit in the weak formulation when γ goes to zero to obtain existence of weak solutions?
- Difficulties:
  - Non smooth not connected fluid domain;
  - Define a functional framework compatible with contact;
  - Test fonctions depend on γ and at the limit on contact point;
  - Compactness of the velocities;

Variational Formulation

$$
\begin{aligned}
&\rho_f \int_{\mathcal{F}_{h_\gamma(t)}} \boldsymbol{u}_\gamma(t)\cdot \boldsymbol{w}_\gamma(t) - \rho_f \int_0^t \int_{\mathcal{F}_{h_\gamma(s)}} \boldsymbol{u}_\gamma \cdot \partial_t \boldsymbol{w}_\gamma + (\boldsymbol{u}_\gamma\cdot\nabla)\boldsymbol{w}_\gamma\cdot \boldsymbol{u}_\gamma \\
&+ \rho_s \int_0^L \partial_t\eta_\gamma(t) b_\gamma(t) - \rho_s \int_0^t\int_0^L \partial_t \eta_\gamma \partial_t b_\gamma + \mu \int_0^t \int_{\mathcal{F}_{h_\gamma(s)}} \nabla \boldsymbol{u}_\gamma : \nabla \boldsymbol{w}_\gamma \\
&+ \beta \int_0^t\int_0^L \partial_x\eta_\gamma\partial_x b_\gamma + \alpha \int_0^t\int_0^L \partial_{xx}\eta_\gamma \partial_{xx} b_\gamma + \gamma \int_0^t\int_0^L \partial_{xt}\eta_\gamma\partial_x b_\gamma \\
&= \rho_f \int_{\mathcal{F}_{h_0}} \boldsymbol{u}^0 \cdot \boldsymbol{w}_\gamma(0) + \rho_s \int_0^L \eta^1 b_\gamma(0).
\end{aligned}
$$

with $\boldsymbol{w}(x, 1+\eta_\gamma(x)) = (0, b_\gamma(x))^T$.

Functional Framework

$$K[h] = \{\boldsymbol{w} \in L^2_\sharp(\Omega), \text{div } \boldsymbol{w} = 0 \text{ in } \Omega, \boldsymbol{w} = 0 \text{ in } (0,L)\times(-1,0), \boldsymbol{w} = b\boldsymbol{e}_2, \text{ for } y > 1+\eta(x)\}$$

$$X[h] = \{(\boldsymbol{w}, b) \in K[h] \times L^2_{\sharp,0}(0,L) \mid w_2|_{(0,L)\times\{M\}} = b\}$$

Compactness

- Build a "regular" displacement $\underline{\eta} \leq \eta_\gamma, \forall\gamma$;
- Obtain compactness on projection of the velocities on $X[\underline{h}]$;
- Build good approximations in $H^\sigma_\sharp(\Omega)\times$ with $\sigma > 0$ of $(\overline{\boldsymbol{w}}_\gamma, b_\gamma)$.

Let $h \in H^2_\sharp(0,L)$ with $h \geq 0$ and $\alpha > 0$.
Define

$$\underline{h}_\alpha := [h-\alpha]_+.$$

Then $\underline{h}_\alpha \in \mathcal{C}^0_\sharp(0,L)$ and, given $\kappa \in (0,1/2)$, there exists a constant C independent of α and h for which:

$$\|\underline{h}_\alpha\|_{H^{1+\kappa}_\sharp(0,L)} + \|\underline{h}_\alpha\|_{W^{1,\infty}_\sharp(0,L)} \leq C\|h\|_{H^2_\sharp(0,L)},$$

$$\|\underline{h}_\alpha - h\|_{W^{1,\infty}_\sharp(0,L)} \leq \alpha + \sup_{\{x\in[0,L] | h(x)\leq\alpha\}} |h'(x)|.$$

Fix $0 < \kappa < \frac{1}{2}$. Let h and $\underline{h}$ belong to $H^{1+\kappa}(0,L) \cap W^{1,\infty}(0,L)$ with $0 \leq \underline{h} \leq h < M-1$ and set

$$A = \|h\|_{H^{1+\kappa}_\sharp(0,L)} + \|h\|_{W^{1,\infty}_\sharp(0,L)} + \|\underline{h}\|_{H^{1+\kappa}_\sharp(0,L)} + \|\underline{h}\|_{W^{1,\infty}_\sharp(0,L)}.$$

Let $s \in [0,\kappa/2)$ and $(\boldsymbol{w},\dot{\eta}) \in X^s[h]$ enjoying the further property:

$$\boldsymbol{w}_{|\mathcal{F}^-_h} \in H^1_\sharp(\Omega^-_h)$$

Then, the following estimate holds true:

$$\|\mathbb{P}^s[\underline{h}](\boldsymbol{w},\dot{\eta}) - (\boldsymbol{w},\dot{\eta})\|_{X^s} \leq C_A(\|\underline{h}-h\|_{W^{1,\infty}_\sharp(0,L)})\|\boldsymbol{w}_{H^1_\sharp(\Omega^-_h)},$$

with $C_A(x) \to 0$ as $x \to 0$.

L^2 compactness

Study the difference

$$
\begin{aligned}
S = \rho_f \int_0^T \int_B |\rho_\gamma \overline{\boldsymbol{u}}_\gamma|^2 + \rho_s \int_0^T \int_0^L |\partial_t \eta_\gamma|^2 - \left(\rho_f \int_0^T \int_B |\rho \overline{\boldsymbol{u}}|^2 + \rho_s \int_0^T \int_0^L |\partial_t \eta|^2 \right) \\
= \sum_k T_1^k + T_3^k - T_3^k
\end{aligned}
$$

with

$$
T_1^k = \int_{I_k} \Big((\rho_\gamma \overline{\boldsymbol{u}}_\gamma, \partial_t \eta_\gamma), (\overline{\boldsymbol{u}}_\gamma, \partial_t \eta_\gamma) - \mathbb{P}^s[\underline{h}_{\delta,k}](\overline{\boldsymbol{u}}_\gamma, \partial_t \eta_\gamma) \Big)_{X^0}
$$

$$
T_2^k = \int_{I_k} \Big((\rho \overline{\boldsymbol{u}}, \partial_t \eta), (\overline{\boldsymbol{u}}, \partial_t \eta) - \mathbb{P}^s[\underline{h}_{\delta,k}](\overline{\boldsymbol{u}}, \partial_t \eta) \Big)_{X^0}
$$

$$
\begin{aligned}
T_3^k = \int_{I_k} \langle \mathbb{P}[\underline{h}_{\delta,k}](\rho_\gamma \overline{\boldsymbol{u}}_\gamma, \partial_t \eta_\gamma), \mathbb{P}^s[\underline{h}_{\delta,k}](\overline{\boldsymbol{u}}_\gamma, \partial_t \eta_\gamma) \rangle_{(X^s)', X^s} \\
- \langle \mathbb{P}[\underline{h}_{\delta,k}](\rho \overline{\boldsymbol{u}}, \partial_t \eta), \mathbb{P}^s[\underline{h}_{\delta,k}](\overline{\boldsymbol{u}}, \partial_t \eta) \rangle_{(X^s)', X^s}
\end{aligned}
$$

Existence results

Rigid Structure:

- Weak solutions:
  - B. Desjardins, M. Esteban: 2D/3D
  - C. Conca, J. San Martin, M. Tucsnak: 2D, one rigid body
  - K.-H. Hoffman, V. Starovoitov: 2D
  - J. San Martin, V. Starovoitov, M. Tuscnak: 2D, contacts
  - M. Gunzburger, H-C. Lee, A. Seregin: 3D, one rigid body

- Strong solutions:
  - C.G, Y. Maday: 2D/3D, small time, inertia of the solid large enough
  - T. Takahashi, M. Tucsnak: 2D/3D, one rigid body

Beam/Plate/Membrane/Shell

- Weak solutions (as long as no collision occur):
  - [Chambolle-Desjardins-Esteban-CG, 05]: Dirichlet boundary conditions, $3D$, $\alpha > 0, \gamma > 0$ and $\delta = \beta = 0$. The proof applies also for $\beta > 0$, $\gamma > 0$, $\delta = \alpha = 0$ in $2D$ case.
    [CG, 09]: Dirichlet boundary conditions, $3D$, $\alpha > 0$ and $\gamma = \delta = \beta = 0$. The proof applies also for $\beta > 0$, $\gamma = \delta = \alpha = 0$ in $2D$ case.
    [Muha-Canic, 13, 15]: $\alpha > 0$, $\gamma \geq 0$, Generalized Neumann boundary conditions;
    [Lengeler-Růžička, 14] : Linear Shell model ($\alpha > 0$);
    [Muha-Schwarzacher] : Non linear Shell model.

- Strong solutions (small time or small data):
  - [Da Veiga, 04], [Lequeurre, 11, 13], [Casanova] : $\gamma > 0$, $\beta = 0$, Dirichlet, periodic boundary conditions or modified Neumann boundary conditions in the 2D case;
  - Steady state case, enclosed cavity: [CG, 98];

Finite number of eigenmodes:

- Weak solutions:
  - B. Desjardins, M. Esteban, C. G., P. Le Tallec: Linearized elasticity, $\mathbf{d}(t,x) = \sum_{i=1}^{N} \alpha_i(t)\phi_i(x)$.
  - M. Boulakia: large displacement–small strain, $\mathbf{u} = \tau + (R - I)\vec{Gx} + R\mathbf{d}$.

- Strong solutions:
  - C. G., Y. Maday, P. Métier: large displacement–small strain, $\mathbf{u} = \tau + (R - I)\vec{Gx} + R\mathbf{d}$, nonlinear model.
  - M. Boulakia, E. Schwindt, T. Takahashi : linearized elasticity, $\mathbf{d}(t,x) = \sum_{i=1}^{N} \alpha_i(t)\phi_i(x)$.

Elastic Solid

- Strong solutions:
  - D. Coutand, S. Shkoller: 3D Saint-Venant Kirschhoff media/ 3D Fluid/Gap of regularity
  - J.-P. Raymond, M. Vanninathan: 3D Saint-Venant Kirschhoff media/ 3D Fluid/ Flat interface
  - M. Boulakia, S. Guerrero, T. Takahashi 3D linearized elastic body in a 3D Fluid

Concluding Remarks

- Can we have contact in the case $\alpha = 0$? γ=0?
- If there is contact do we have uniqueness of the solution?
- Generalization to other boundary conditions at the outlets?
- Consider the longitudinal displacement of the elastic structure.
- What about the 3D case?