NOTES ON COMPACT MODULI OF K3 SURFACES

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Introduction: Beginnings of moduli

0.1. **Early history.** The study of a moduli space of Riemann surfaces of genus g was initiated (unsurprisingly) by Riemann, who was the first to perform the heuristic calculation that the space of such surfaces depends on $3g-3$ complex parameters, or "moduli."

In the century following, the beautiful theory of the moduli space $\mathcal{M}_g$ was uncovered by work of Klein, Poincaré, Teichmüller, and others. The development of this theory played an integral role in many fields of mathematics, for instance in introducing formal notions of topological spaces, manifolds, and groups. Understanding Riemann surfaces was a major motivation of Klein's *Erlangen program*, which sought to understand geometry in terms of the group of symmetries of those geometries.

Study of K3 surfaces, while somewhat more recent than that of curves, also spurred many important developments in mathematics, such as Hodge theory and the mimimal model program. It was initiated by the Italian school of algebraic geometry, in the early 20th century. The Erlangen program had already found that algebraic curves split into three broad categories: The positively curved case $g=0$, the flat case $g=1$, and the negatively curved case $g\geq 2$.

Enriques, Castelnuovo, and later Kodaira, extended these results to surfaces, categorizing them by their *Kodaira dimension*

$$\kappa(X) := -1 + \dim \textstyle\bigoplus_{m\geq 0} H^0(X, mK_X).$$

The $\kappa=-\infty$ surfaces are ruled, the $\kappa=0$ surfaces are Calabi-Yau, the $\kappa=1$ surfaces are elliptically fibered, and the remaining "general type" surfaces have $\kappa=2$. Within the $\kappa=0$ surfaces are those covered by an abelian surface (the abelian and bielliptic surfaces), and those covered by a K3 surface (the Enriques and K3 surfaces).

Definition 0.1. A *K3 surface* X is a compact complex surface, which is simply connected and has trivial canonical bundle $K_X=\mathcal{O}_X$.

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Example 0.2. Let $X \subset \mathbb{P}^3$ be a smooth hypersurface of degree 4. The adjunction formula implies that $K_X = (K_{\mathbb{P}^3} + 4H)\big|_X = \mathcal{O}_X$. By Lefschetz hyperplane theorem, $\pi_1(X) = \pi_1(\mathbb{P}^3) = 0$ so X is a K3 surface. Other examples are the complete intersections $X_{2,3} \subset \mathbb{P}^4$, $X_{2,2,2} \subset \mathbb{P}^5$, and double covers $X \to \mathbb{P}^2$ branched over a sextic curve.

Some of the earliest K3 surfaces considered were the "Flächen vierten Grades mit sechzehn singulären Punkten" of Kummer in 1884. These are the quotients $A/\{\pm 1\}$ of abelian surfaces A by negation.

Theorem 0.3 (Enriques 1909)**.** *For all $g \geq 3$, there are surfaces $X \subset \mathbb{P}^g$ of degree $2g-2$ embedded by a complete linear system, with trivial canonical bundle $K_X = \mathcal{O}_X$ and $h^1(X, \mathcal{O}_X) = 0$.*

Theorem 0.4 (Severi 1909)**.** *For each $2d = 2g-2$, there are 19 moduli of such surfaces.*

Example 0.5. Counting parameters for quartic hypersurfaces, we have the space of quartic polynomials on $\mathbb{P}^3$, which has dimension $\binom{7}{4} = 35$, minus the space of linear transformations $\mathrm{GL}_4(\mathbb{C})$, which has dimension 16. Thus, the parameter count is $35 - 16 = 19$.

Definition 0.6. A *polarized K3 surface* (X, L) is a K3 surface together with an ample line bundle $L \to X$.

The term *K3 surface* was coined by Weil in 1958, who named them after the three mathematicians: Kähler, Kummer, Kodaira and after the K2 mountain (Weil appreciated its beauty). One of the first major results concerning the classification of K3 surfaces arose from the work of Kodaira and Kuranishi, who developed the theory of deformations of complex structure.

Theorem 0.7 (Kodaira 1964)**.** *All K3 surfaces are deformation equivalent, and the space of complex deformations of a K3 surface is 20-dimensional.*

Thus the 19-dimensional families of polarized K3 surfaces are contained in a single 20-dimensional family of analytic K3 surfaces.

We can also conclude that all K3 surfaces are diffeomorphic, and more weakly, have the same cohomology ring. We have $H^i(X, \mathbb{Z}) = 0$ for $i = 1, 3$, $H^i(X, \mathbb{Z}) = \mathbb{Z}$ for $i = 0, 4$. Most important is $H^2(X, \mathbb{Z}) \simeq \mathbb{Z}^{22}$. There is a symmetric perfect pairing on $H^2(X, \mathbb{Z})$ making it isometric to the even unimodular lattice $II_{3,19}$ of signature $(3, 19)$.

Importantly, the second cohomology $H^2(X, \mathbb{Z})$ admits a weight 2 *Hodge structure*: Upon tensoring with $\mathbb{C}$, we get a decomposition

$$H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

for which $H^{p,q} = \overline{H^{q,p}}$. The space $H^{p,q}(X)$ is represented by the harmonic (p,q)-forms. We additionally have $(H^{2,0})^{\perp} = H^{2,0} \oplus H^{1,1}$ and $x \cdot \bar{x} > 0$ for any nonzero $x \in H^{2,0} = H^0(X, \Omega^2) \simeq \mathbb{C}$.

Theorem 0.8 (Siu 1983). *All K3 surfaces are Kähler.*

0.2. **Torelli theorems.** An important development in the theory of moduli of curves was the result of Torelli that a curve could be reconstructed from essentially linear-algebraic data.

Theorem 0.9 (Torelli, 1913). *Let $\{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}$ be a standard system of curves on a Riemann surface C and let $(\omega_1, \dots, \omega_g)$ be the basis of the abelian differentials on C for which $\int_{\alpha_i} \omega_j = \delta_{ij}$. Then the isomorphism type of C is uniquely recoverable from the symmetric $g \times g$ period matrix $(\int_{\beta_i} \omega_j)$.*

In modern terminology, we would say: C can be recovered from the polarized Hodge structure on $H^1(C, \mathbb{Z})$. The central result to the understanding of moduli of K3 surfaces is provided by an analogous "Torelli theorem" of Piatetski-Shapiro and Shafarevich:

Theorem 0.10 (Piatetski-Shapiro, Shafarevich 1973). *Two K3 surfaces X and X' are isomorphic if and only if there exists an isometry*

$$\phi\colon H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$$

for which $\phi(H^{2,0}(X)) = H^{2,0}(X')$.

We call such an isometry a *Hodge isometry*. In addition to this statement concerning the isomorphism type of a single surface, we also have a local Torelli theorem: The complex deformation space of X is locally isomorphic to the period space:

Definition 0.11. The *period domain* of K3 surface is

$$\mathbb{D} := \mathbb{P}\{x \in II_{3,19} \otimes \mathbb{C} \,|\, x \cdot x = 0, x \cdot \bar{x} > 0\}.$$

A *marking* of a K3 surface is an isometry $\phi\colon H^2(X, \mathbb{Z}) \to II_{3,19}$ and the *period* of a marked K3 surface (X, ϕ) is $\phi(H^{2,0}(X)) \in \mathbb{D}$.

Theorem 0.12 (Local Torelli Theorem). *Let (X, ϕ) be a marked K3 surface and let $\mathfrak{X} \to U \subset H^1(X, T_X) \simeq \mathbb{C}^{20}$ be the Kuranishi deformation. With a choice of marking of the fibers of $\mathfrak{X} \to U$, the resulting period map $U \to \mathbb{D}$ is a local isomorphism.*

Let $L \to X$ be a primitive ample line bundle. Then $c_1(L)\cdot[\Omega] = 0$ and furthermore, $c_1(L) \in H^2(X, \mathbb{Z})$. We can refine the notion of a marking for a pair (X, L) by choosing $v \in II_{3,19}$ primitive with $v \cdot v = 2d$ and require a marking of (X, L) to send $\phi\colon c_1(L) \mapsto v$. Then the period mapping lands rather in

$$\mathbb{D}_{2d} = \mathbb{P}\{x \in v^\perp \otimes \mathbb{C} \,\big|\, x \cdot x = 0, x \cdot \bar{x} > 0\}.$$

Let $\Gamma_{2d} := \{\gamma \in O(II_{3,19}) \,\big|\, \gamma(v) = v\}$. The great upshot of restricting to polarized K3 surfaces is that now Γ_{2d} acts properly discontinuously on $\mathbb{D}_{2d}$. Hence, given any family $(\mathfrak{X}, \mathfrak{L}) \to S$ over a base S, we have a *period map* $S \to \Gamma_{2d}\backslash\mathbb{D}_{2d}$.

Theorem 0.13. *The period map identifies the moduli space of polarized smooth K3 surfaces of degree* $2d$ *with a Zariski open subset of* $\Gamma_{2d}\backslash\mathbb{D}_{2d}$. *If we allow the polarized K3 surface to have ADE singularities, then the moduli space of degree* $2d$ *K3 surfaces is exactly* $F_{2d} := \Gamma_{2d}\backslash\mathbb{D}_{2d}$.

Additionally, we have the following well-known theorem:

Theorem 0.14 (Baily-Borel 1966)**.** *The quotient* $\Gamma_{2d}\backslash\mathbb{D}_{2d}$ *is a quasiprojective variety.*

0.3. **Compactification of moduli.** The compactification of moduli spaces also has its origins in $\mathcal{M}_g$. A landmark achievement in algebraic geometry was Deligne and Mumford's the construction of a compactification

$$\mathcal{M} \hookrightarrow \overline{\mathcal{M}}_g$$

at least for $g \geq 2$. There are numerous reasons why a compactification is useful. One is to compute intersection numbers, which in general will not be well-defined unless one has a compact parameter space.

Example 0.15. Given a generic pencil of plane curves of degree d, how many have an effective even theta characteristic? The answer is an intersection number between a curve and a divisor in some $\overline{\mathcal{M}}_g$.

For the best possibility of application, we would like a compactification to enjoy the following two properties:

(1) The compactification should be *modular*: It should parameterize a generalization of the geometric objects parameterized in the original moduli space.
(2) The compactification should be smooth, or at least mildly singular, with understood boundary combinatorics.

The second point is important for intersection theory. One needs, in practice, an explicit understanding of the boundary strata to perform computations in the compactification.

We have the best possible situation for $\overline{\mathcal{M}}_g$: (1) this compactification parameterizes stable curves generalizing smooth curves, and (2) $\overline{\mathcal{M}}_g$ has normal crossings boundary, and the boundary strata are completely explicit, corresponding to stable graphs of genus g. Another result analogous to this is for abelian varieties:

Theorem 0.16 (Mumford 1976, Namikawa 1980, Alexeev 1996)**.** *The moduli space $\mathcal{A}_g$ of principally polarized abelian varieties (PPAVs) admits a toroidal compactification with a modular interpretation, called the "second Voronoi compactification".*

It has toroidal singularities, its strata are in bijection with Voronoi decompositions of lattices in $\mathbb{R}^h$, $h \leq g$, and the boundary points parameterize a generalization of a PPAV. In particular, the universal family and the universal theta divisor extend over the boundary.

Thus, we pose the following question, which we will try to answer in the following three lectures:

Question 0.17. Is there an analogous compactification of F_{2d}? One which is both modular and has toroidal singularities?

1. Degenerations of K3 surfaces

1.1. **Kulikov models.** To understand what to put at the boundary of moduli, we must begin by considering one parameter degenerations. The valuative criterion of properness implies that any map $C^* \to F_{2d}$ from a punctured curve $C^* = C \setminus 0$ to the moduli space admits a unique extension $C \to \overline{F}_{2d}$ to the compactification. Thus, if $\overline{F}_{2d}$ is to parameterize some geometric objects, a central question, almost synonymous with the compactification question, is: *How do I extend the family of polarized K3 surfaces $(X^*, L^*) \to C^*$ uniquely over the puncture?*

As we well know from the example of $\mathcal{M}_g$, the moduli space may only end up as an orbifold/Deligne-Mumford stack. This means we should allow finite base changes to the curve C^*. Thus, our starting point is the semistable reduction theorem, due to Mumford:

Theorem 1.1. *Let $X^* \to C^*$ be a family of smooth projective varieties with smooth total space. After a finite base change, there is an extension $X \to (C, 0)$ such that*

(1) *X is smooth, and*
(2) *X_0 has simple normal crossings.*

We call $X \to C$ a *semistable model.* Recall that X_0 has *simple normal crossings* (is SNC), if the scheme-theoretic fiber of $0 \in C$ is analytically locally cut out by an equation of the form $x_1 \cdots x_k = 0$ in an analytic coordinate patch $(x_1, \dots, x_n)$ on the smooth complex manifold X. For the more algebraically minded reader, it suffices to work in the étale topology.

One can wonder whether this is the possible. After all, semistable models are highly non-unique. For instance, one can simply blow up a smooth point of X_0 in the total space X and produce a new semistable model. The starting point for K3 compactification is a stronger form of the semistable reduction theorem.

Theorem 1.2 (Kulikov 1977, Persson-Pinkham 1981)**.** *Let $X^* \to C^*$ be a degeneration of K3 surfaces. After a finite base change, there is a semistable model $X \to (C, 0)$ which additionally satisfies:*

(3) *$K_X \sim_C \mathcal{O}_X$.*

We call such a filling a *Kulikov model.* If Kulikov models were unique for any given degeneration, we would essentially be done, but this is not the case.

Example 1.3. Let $\lambda(x_0^4 + x_1^4 + x_2^4 + x_3^4) + x_0x_1x_2x_3 = 0$ be the Fermat degeneration of quartic hypersurfaces in $\mathbb{P}^3$ over the curve with coordinate λ. At $\lambda = 0$, we get a singular surface, the union of the coordinate hyperplanes in $\mathbb{P}^3$. Thus X_0 is SNC. But, this is not a Kulikov model, because the total space is singular. There are 24 singular points, at

$$V(x_i,\, x_j,\, x_0^4 + x_1^4 + x_2^4 + x_3^4) \text{ for } i < j$$

and a local analytic model of the singularity is $V(ab - cd) \subset \mathbb{C}^4$.

This singularity is called the *conifold*, the simplest and most important singularity of a threefold. Blowing up $(0,0,0,0) \in V(ab-cd)$ gives an exceptional divisor which is the projective quadric $V(ab-cd) \subset \mathbb{P}^3$, isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. This exceptional surface may be contracted along either ruling of $\mathbb{P}^1 \times \mathbb{P}^1$ and still give a smooth threefold. These then give the two *small resolutions* of the conifold—resolutions with no divisorial exceptional locus.

For one conifold point $p \in V(x_i,\, x_j,\, x_0^4 + x_1^4 + x_2^4 + x_3^4)$, there are thus two resolutions, either one replaces p with $\mathbb{P}^1$. But the two different conifold resolutions produce visibly different surfaces. In one resolution, the exceptional curve lies on the component $V(x_i)$ while in the other resolution, it lies on $V(x_j)$. Thus, the Fermat degeneration admits 2^{24} conifold resolutions, all of them Kulikov models.

Exercise 1.4. Model the conifold $V(ab - cd)$ as a toric threefold, and determine its fan explicitly. How can you describe the two resolutions in term of toric geometry and the fan?

In fact, the *conifold transition* or *Atiyah flop* between the two resolutions is the only ambiguity in Kulikov models:

Theorem 1.5 (Shepherd-Barron, 1983)**.** *Any two Kulikov models $X \to (C, 0)$ and $X' \to (C, 0)$ extending a degeneration $X^* \to C^*$ can be connected by a sequence of Atiyah flops along $\mathbb{P}^1$-curves in the central fiber with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.*

To illustrate the centrality of K3 surfaces to the development of mathematics: The minimal model program originated with an attempt to understand how to relate different Kulikov models.

1.2. **Kulikov models, explicitly.** What does a Kulikov model look like? It is important for us to have a concrete understanding of the central fiber X_0. We have already seen that for the Fermat family, the surface X_0 results from gluing together various blow-ups of $\mathbb{P}^2$ along triangles of curves. Thus X_0 is, combinatorially, a tetrahedron. Kulikov models naturally divide into three types:

Definition 1.6. We say that $X \to (C, 0)$ is *Type I* if X_0 is smooth, *Type II* if X_0 a double locus but no triple points, and *Type III* if X_0 has triple points.

In other words $X \to (C, 0)$ is Type I if for X_0 we only need the local analytic model $x = 0$, Type II if we only need $x = 0$, $xy = 0$, and Type III if we need $x = 0$, $xy = 0$, $xyz = 0$.

Theorem 1.7 (Description of Kulikov models)**.** *A Type I degeneration X_0 is a smooth K3 surface.*

A Type II degeneration $X_0 = \bigcup_{i=0}^{n} V_i$ is a chain of surfaces with V_0 and V_n rational, and V_i for $i \in \{1, \ldots, n-1\}$ birational to $E \times \mathbb{P}^1$ for E a fixed elliptic curve. Each component $D_{i,i+1} = V_i \cap V_{i+1}$ is a copy of the elliptic curve E, and furthermore the double locus is an anticanonical divisor on V_i. We have

$$D_{i,i+1}\big|_{V_i}^2 + D_{i,i+1}\big|_{V_{i+1}}^2 = 0.$$

A Type III degeneration $X_0 = \bigcup V_i$ is a union of rational surfaces. Setting $D_{ij} = V_i \cap V_j$, the double locus $\sum_j D_{ij} \subset V_i$ is an anticanonical cycle of rational curves on V_i. The double curves satisfy

$$D_{ij}\big|_{V_i}^2 + D_{ij}\big|_{V_j}^2 = -2.$$

Finally, the combinatorial arrangement of the surfaces V_i is a two-dimensional sphere S^2.

Exercise 1.8. Prove the statements concerning D_{ij}^2 and the canonical bundles of V_i using adjunction formula and that fact that $K_X = \mathcal{O}_X$.

To be more precise about the "combinatorial arrangement": Given any SNC degeneration $X \to (C, 0)$, we can build a simplicial complex $\Gamma(X_0)$ called the *dual complex* as follows:

(1) the 0-simplices v_i correspond to components $V_i \subset X_0$,
(2) the 1-simplices e_{ij} correspond to double loci $D_{ij} = V_i \cap V_j$,
(3) the 2-simplices t_{ijk} correspond to triple loci $T_{ijk} = V_i \cap V_j \cap V_k$,

etc. Thus, the dual complex of a Type I degeneration is a point, the dual complex of a Type II degeneration is a segment, broken up into subsegments, and finally the dual complex of a Type III degeneration is a triangulation of S^2.

See Figure 6 for a heuristic diagram of a Type III degeneration: One depicts the components V_i as 2-cells, the double curves D_{ij} as edges, and the triple points T_{ijk} as trivalent vertices of the diagram. Figure 5 above depicts part of the dual complex, a triangulation of the sphere.

Remark 1.9. Conjecturally, something like this holds for all Calabi-Yau degenerations: For "maximal" degenerations of:

(1) abelian varieties, $\Gamma(X_0)$ will be homeomorphic to $(S^1)^n$,
(2) hyperkähler varieties, $\Gamma(X_0)$ will be homeomorphic to $\mathbb{CP}^n$,

(3) strict Calabi-Yau varieties, $\Gamma(X_0)$ will be homeomorphic to S^n.

All Calabi-Yau varieties are, up to an étale cover, a product of such, and thus, simplicial decompositions of products of tori, $\mathbb{CP}^n$'s, and S^n's are expected to govern the combinatorics of these degenerations.

The only overlaps are $S^1 = S^1$, corresponding to elliptic curves, and $S^2 = \mathbb{CP}^1$, corresponding to K3s.

1.3. **Anticanonical pairs.** We would like to understand the components of a Kulikov model explicitly. Our primary focus is the Type III case, which is the most interesting and complicated, on both a moduli-theoretic and combinatorial level.

Definition 1.10. An *anticanonical pair* (V, D) is a smooth rational surface V together with an anticanonical divisor $D \in |-K_V|$ which is a cycle/wheel $D = D_1 + \cdots + D_n$ of rational curves.

The most important example of an anticanonical pair is a smooth toric surface V with its toric boundary as D.

Crash Course on Toric Surfaces 1.11. Smooth, complete, toric surfaces are, up to the natural action of $GL_2(\mathbb{Z})$, in bijection with cycles of vectors $\vec{e}_i \in \mathbb{Z}^2$ for which $\vec{e}_i$ and $\vec{e}_{i+1}$ form an oriented lattice basis of $\mathbb{Z}^2$ (and which wind only once around the origin). For example, the cyclic sequence of vectors $(1,0), (0,1), (-1,-1)$ successively form lattice bases, and correspond to the toric surface $\mathbb{P}^2$. The most important facts about toric surfaces for us are:

(1) The vectors $\vec{e}_i$ are in bijection with the components D_i of the toric boundary, i.e. the 1-dimensional torus orbits.
(2) There is a combinatorial formula for the self-intersection number $D_i \cdot D_i = D_i^2$ given by
$$\vec{e}_{i-1} + \vec{e}_{i+1} = (-D_i^2)\vec{e}_i.$$

For instance, in the above example, this determines that the three components of the toric boundary of $\mathbb{P}^2$ each have self-intersection 1.

Exercise 1.12. Prove that $12 + \sum(-D_i^2 - 3) = 0$ for any toric surface.

Example 1.13.

(1) $(\mathbb{P}^2, L_1 + L_2 + L_3)$ with an anticanonical triangle of lines,
(2) $(\mathbb{P}^2, L + C)$ with a transversely intersecting line and conic,
(3) $(\mathbb{P}^1 \times \mathbb{P}^1, s_1 + f_1 + s_2 + f_2)$ with a anticanonical square formed from two sections and two fibers,

(4) the blowup $Bl_p(V, D)$ of an anticanonical pair at a smooth point $p \in D$, with anticanonical divisor the strict transform of D,
(5) the blowup $Bl_p(V, D)$ of an anticanonical pair at a node $p \in D$, with anticanonical divisor the reduced inverse image of D.

In fact, using Examples (4) and (5), called *interior* and *corner* blowups, respectively, along with the reverse blow-down operations, one can pass between any two anticanonical pairs. Examples (1) and (3) are toric, and (5) maintains toricity, while (4) always destroys toricity. Note that, even though $\mathbb{P}^2$ is a toric surface, Example (2) is not toric because its anticanonical cycle is not the toric boundary.

Definition 1.14. The *charge* of an anticanonical pair is $Q(V, D) := 12 + \sum(-D_i^2 - 3)$, if D has at least two components. If D is irreducible, we instead define $Q(V, D) := 11 - D^2$.

The following theorem, while easy to prove, presaged the existence of an important geometric structure on $\Gamma(X_0)$, discovered by mirror-symmeters (cf. Hitchin, Kontsevich-Soibelman, Gross-Siebert, Gross-Hacking-Keel).

Theorem 1.15 (Friedman-Miranda, 1983)**.** *Let X_0 be a Type III Kulikov model. Then the sum of the charges is*

$$\textstyle\sum_i Q(V_i, \sum_j D_{ij}) = 24.$$

Exercise 1.16. Prove this formula, using Gauss-Bonnet, the fact that $\Gamma(X_0)$ is a triangulation, and the formula $D_{ij}^2 + D_{ji}^2 = -2$.

In particular, there are at most 24 non-toric components of a Type III Kulikov model, since $Q(V, D) \geq 0$ with equality iff (V, D) is toric.

1.4. **The integral-affine structure on the dual complex.** Let us pay particular attention to the formula

$$\vec{e}_{i-1} + \vec{e}_{i+1} = (-D_i^2)\vec{e}_i \tag{1.4.1}$$

from the toric geometry of surfaces.

Question 1.17. If we try to lay the directed edges $\vec{e}_{ij}$ emanating from a vertex $v_i \in \Gamma(X_0)$ onto vectors in $\mathbb{R}^2$, then can we ensure that Equation (1.4.1) always holds, around every vertex?

If v_i corresponds to a toric component $(V_i, \sum_j D_{ij})$ of X_0, then near v_i this is possible by placing v_i at a lattice point in $\mathbb{Z}^2$ and placing the edges $\vec{e}_{ij}$ as the primitive integral vectors defining the fan of V_i.

As we expand out from a toric vertex v_i continuing to impose that Equation (1.4.1) holds, we find, intriguingly, that the fans patch together to form a triangular quilting of the plane, see Figure 1.

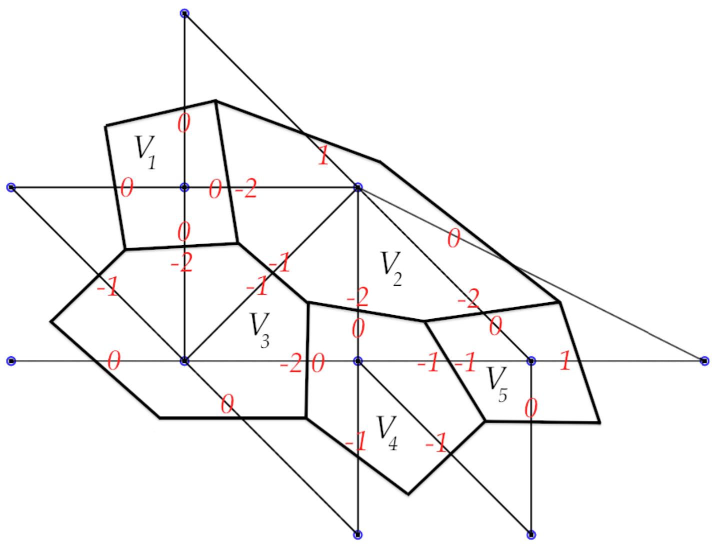

FIGURE 1. Gluing fans together so Equation (1.4.1) holds around multiple vertices of $\Gamma(X_0)$.

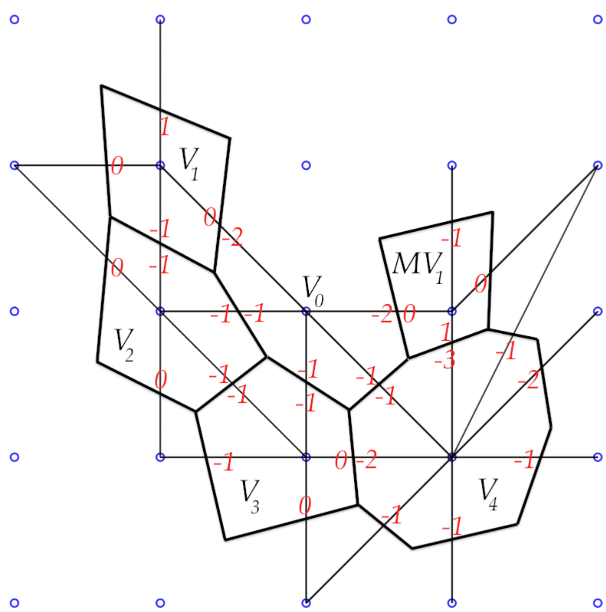

FIGURE 2. Monodromy around nontoric vertices.

But sometimes this fails. For example, suppose we have a component $V_0 \subset X_0$ whose anticanonical cycle $D_0 \subset V_0$ consists of four curves, of self-intersection numbers $(-2, -1, -1, -1)$. This is easily constructed, for instance, by taking V_0 as the result of 5 internal blowups on (3) in Example 1.13.

One finds, that, as one tries to patch together the fans of the adjacent surfaces, they fail to form a quilting of the plane. The issue is shown in Figure 2. It is as though going around the vertex v_0 in the dual complex has a "monodromy". This monodromy is exactly due to the non-toricity of the component V_0.

Answer 1.18. We can only endow $\Gamma(X_0)$ with a local embedding into $\mathbb{R}^2$ satisfying Equation (1.4.1). Thus, we can think of $\Gamma(X_0)$ as only having a locally flat structure, which undergoes a monodromy transformation as one goes around non-toric vertices of the dual complex.

More precisely:

Theorem 1.19 (Gross-Hacking-Keel 2011, Engel 2014)**.** *The dual complex $\Gamma(X_0)$ of a Type III Kulikov model admits canonically the structure of an integral-affine sphere, or* IAS^2*, with up to 24 singularities: the open subset*

$$\Gamma(X_0)\setminus\{v_i\,\big|\,(V_i,\textstyle\sum_j D_{ij})\textit{ is non-toric}\}$$

admits a collection of charts to $\mathbb{R}^2$ whose transition functions lie in the integral-affine transformation group $SL_2(\mathbb{Z})\ltimes\mathbb{Z}^2$.

The local embedding into $\mathbb{R}^2$ sends any triangle in $\Gamma(X_0)$ to a lattice triangle of area $\frac{1}{2}$. Around any toric vertex v_i the local embedding gives the fan of the component V_i with the edges $\vec{e}_{ij}$ emanating from v_i giving the primitive integral vectors of the fan.

1.5. **Polarized** IAS^2.

The above discussion of Kulikov models, and the combinatorics of the dual complexes has, so far, ignored the presence of a polarization L^* on the punctured family $X^*\to C^*$.

Theorem 1.20 (Shepherd-Barron, 1983)**.** *A Kulikov model can be chosen $X\to(C,0)$ for which the line bundle $L^*\to X^*$ extends to a line bundle $L\to X$ which is relatively big and nef over C. Then $|4L|$ defines a contraction of the curves in fibers intersecting L to be zero.*

Definition 1.21. We call $(X,L)\to(C,0)$ a *nef model.*

One might hope that the nef model of a degeneration is unique, but again, this is false. More importantly, the image of $|4L|$ is also not unique, so that need not be the limiting object in moduli either. This is due to the fact that there are many different extensions of $L^*\to X^*$ to a line bundle on $L\to X$. The space of such extensions is a torsor over the subgroup of $\mathrm{Pic}(X)$ generated by $\mathcal{O}_X(V_i)$.

Suppose we are given a nef model. From it, we can produce a collection of non-negative integers, one for each edge of the dual complex:

$$n_{ij} := \deg(L\big|_{D_{ij}}).$$

Crash Course on Toric Surfaces 1.22. Let (V, D) be a smooth, complete toric surface associated to the sequence of vectors $\vec{e}_i \in \mathbb{Z}^2$. Under what conditions does there exist a line bundle $L \to V$ for which $L \cdot D_i = n_i$? And under what circumstances is this line bundle nef?

The answer both questions is quite simple. For the first question, we need that $\sum n_i \vec{e}_i = 0$ is the zero-vector. For the second question, we need $n_i \geq 0$. Thus, we can think of a nef line bundle on V as a weighted balanced graph on its fan. The graph is supported on the vectors $\vec{e}_i$, the weight is n_i and the balancing condition is given by $\sum n_i \vec{e}_i = 0$.

For example, a graph with positive weights n_1, n_2, n_3 on the rays $(1, 0)$, $(0, 1)$, $(-1, -1)$ will be balanced if and only if $n_1 = n_2 = n_3$ in which case the line bundle $\mathcal{O}_{\mathbb{P}^2}(n_1)$ has the desired intersection numbers with the three toric boundary components.

We now state an extension of Theorem 1.19:

Theorem 1.23 (Alexeev-Engel-Thompson, 2019, Alexeev-Brunyate-Engel 2022)**.** *Let $(X, L) \to (C, 0)$ be a nef model. Then $\Gamma(X_0)$ admits the structure of a* polarized IAS^2*: A triangulated integral-affine sphere, together with an effective, weighted, balanced graph $L_{\mathrm{IA}} \subset \Gamma(X_0)$. With appropriate definitions, the converse holds too.*

We only analyzed the "balancing condition" in 1.22 for a toric vertex $v_i \in \Gamma(X_0)$ and a modified definition of balancing is required for the non-toric vertices.

2. Compactification of the moduli space

2.1. **KSBA compactifications.** We will begin by understanding the question of *modular* compactifications, i.e. compactifications which parameterize some generalization of a K3 surface.

It is instructive to look at curves first. Recall that we have a moduli space $\mathcal{M}_{g,n}$ of genus g curves with n marked points, and a compactification $\overline{\mathcal{M}}_{g,n}$ of it, *but only when the inequality* $3g-3+n>0$ holds. There are two ways to view this inequality:

(1) It imposes the condition that $\mathrm{Aut}(C, x_1, \dots, x_n)$ is finite.
(2) It imposes the condition that $K_C + (x_1 + \cdots + x_n)$ is ample.

It is clear why we would want to impose (1) as it is necessary condition to ensure that our moduli forms a Deligne-Mumford stack/orbifold, but it is condition (2) which generalizes best to higher dimensions.

Let us also observe the following property of marked stable curves $(C, x_1, \dots, x_n) \in \overline{\mathcal{M}}_{g,n}$: The singularities of C are the simplest possible curve singularities (simple nodes) and the marked points x_i are never nodes of C.

Of course, there are degenerations of curves which do not satisfy these properties. For instance, one can degenerate a family of elliptic curves in $\mathcal{M}_{1,1}$ to a cuspidal curve. But it is *always possible to replace such a degeneration with one satisfying the above conditions, because* $\overline{\mathcal{M}}_{g,n}$ *is proper.*

A landmark achievement of MMP was to generalize this result to higher dimensions. We require the following definition:

Definition 2.1. Let (X, Δ) be a projective variety X together with an effective $\mathbb{Q}$-Weil divisor Δ. We say that the pair is *slc stable* if:

(1) $K_X + \Delta$ is $\mathbb{Q}$-Cartier and ample.
(2) (X, Δ) has semi-log canonical (slc) singularities.

The definition of slc singularities is somewhat technical. An approximate definition is that if one takes an SNC resolution $\pi\colon (\widetilde{X}, \widetilde{\Delta}) \to (X, \Delta)$, then the "log discrepancy divisor"

$$(K_{\widetilde{X}} + \widetilde{\Delta}) - \pi^*(K_X + \Delta),$$

necessarily a linear combination of the exceptional divisors of π, has all coefficients ≥ -1. This definition has to be modified if (X, Δ) is non-normal to include the double locus as part of Δ on each component of the normalization.

Exercise 2.2. Let X be a curve and suppose that Δ is a finite sum of points with all coefficients 1. Check that (X, Δ) is slc stable if and only if X is nodal and the points in Δ are not nodes of X or equal to each other.

Theorem 2.3 (Kollar-Shepherd-Barron 1988, Alexeev 1996, Kovacs–Patakfalvi 2017)**.** *The components of moduli of slc stable varieties are projective.*

In particular, any degenerating family of slc stable varieties has a unique slc stable limit. This is a stunning generalization of $\overline{\mathcal{M}}_{g,n}$ to moduli of varieties in all dimensions. Unfortunately, very little is known about the actual projective moduli spaces which arise, and "Murphy's Law" is expected to hold—they are likely to have singularities as bad as you can imagine.

2.2. **Choices of polarizing divisor.** An issue arises for K3 surfaces: If we take the divisor Δ to be empty, then $K_X + \Delta = \mathcal{O}_X$ is not ample!

The same issue arises for genus 1 curves (and abelian varieties) and we do not have a nice moduli space $\mathcal{M}_1$. Rather, we must choose an origin $x \in C$ and then we get the space $\mathcal{M}_{1,1}$ which admits a nice compactification. But a choice of origin on a genus 1 is, in a sense, no choice at all, because there is an isomorphism between the two curves (C, x) and (C, x') with marked point, for any two choices of origin x, x'.

We will employ the same principle for polarized K3 surfaces, by choosing an ample divisor on (X, L) which is canonically determined by (X, L) itself, so that any isomorphism between two polarized K3 surfaces identifies the corresponding divisors.

Definition 2.4. Let $(\mathfrak{X}, \mathfrak{L}) \to F_{2d}$ be the universal polarized K3 surface and let $\mathfrak{X}_{\mathbb{C}(F_{2d})}$ be the generic polarized K3 surface, i.e. the K3 surface over the generic point of the moduli space. A *canonical choice of polarizing divisor* is a section $\mathfrak{R} \in |n\mathfrak{L}|$.

Alternatively, a canonical choice of polarizing divisor is an algebraically varying divisor in $|nL|$, over a Zariski-open subset $U \subset F_{2d}$.

Example 2.5. Consider the moduli space F_2 of degree 2 K3 surfaces (X, L). Over a Zariski-open subset of F_2, we have that

$$\phi_{|L|} \colon X \to \mathbb{P}^2$$

is a double cover, branched over a smooth sextic curve. The ramification divisor $R \in |3L|$ is a canonical choice of polarizing divisor.

Example 2.6. Consider the moduli space F_4 of degree 4 K3 surfaces (X, L). Over a Zariski-open subset of F_4 we have that $X = V(f_4) \subset \mathbb{P}^3$ is the vanishing of a smooth quartic. Let

$$\mathrm{Hess}(f) := \det(\tfrac{\partial^2 f}{\partial x_i \partial x_j})$$

be the determinant of the Hessian matrix of second partial derivatives. Then $R = V(\mathrm{Hess}(f)) \cap X \in |8L|$ is a canonical choice of polarizing divisor for F_4.

Example 2.7. Consider the moduli space F_{2d} of degree $2d$ K3 surfaces (X, L). Xi Chen proved that, there are exactly n_d curves $\{R_i\}_{i=1}^{n_d}$ of geometric genus zero in $|L|$ over a Zariski-open subset of F_{2d}. A famous formula due to Yau-Zaslow states that $n_d = [q^d]\,\Delta(q)^{-1}$ where $\Delta(q)$ is the modular discriminant. Then $R := \sum R_i \in |n_d L|$ is a canonical choice of polarizing divisor, which we call the *rational curve divisor*.

Given a canonical choice of polarizing divisor, the pairs $(X, \epsilon R)$ now *are* slc stable, for all sufficiently small epsilon! this gives us a method of compactifying the moduli space F_{2d}.

Definition 2.8. Let $\overline{F}_{2d}^R$ be the closure of the space of pairs $(X, \epsilon R)$ in the projective moduli space of all slc stable pairs.

This gives a modular compactification of the open subset $U \subset F_{2d}$ since by construction, the moduli of pairs $(X, \epsilon R)$ extends to the boundary. But can we control the geometry of this compactification?

2.3. **Divisor models.** How, in practice, does one compute an slc stable limit of pairs $(X, \epsilon R)$? One answer is due to Laza:

Theorem 2.9 (Laza 2016, Alexeev-Engel-Thompson 2019)**.** *Consider a degenerating family $(X^*, \epsilon R^*) \to C^*$ of slc stable K3 pairs. Then, up to a finite base change, there is a Kulikov model $(X, \epsilon R) \to (C, 0)$ for which $R \supset R^*$ is a flat extension of the divisor R^* enjoying the following properties:*

1. *R_0 is nef,*
2. *R_0 contains no strata of X_0.*

Thus, we can view this as a refinement of Shepherd-Barron's Theorem 1.20 on the existence of an extension of (X^*, L^*) so that L is relatively nef. For instance, $\mathcal{O}_X(R)$ will be such an extension.

Definition 2.10. We call $(X, \epsilon R)$ a *divisor model.* Let

$$\overline{X} = \mathrm{Proj}_C \bigoplus_{n \geq 0} \pi_*(\mathcal{O}_X(nR))$$

and let $\overline{R}$ be the image of R under the contraction $X \to \overline{X}$. We call $(\overline{X}, \epsilon\overline{R})$ be the *stable model.*

This is the unique stable limit of the pairs $(X^*, \epsilon R^*)$ in slc stable moduli. To summarize: The method of computing a stable limit is to first find a Kulikov model X_0, perform sequences of Atiyah flops until the flat limit R_0 of the divisors R_t satisfy properties (1) and (2), and finally contract all curves intersecting R_0 to be zero.

2.4. **Toroidal compactifications.** We now discuss compactifications of F_{2d} of an a priori entirely different nature. These compactifications exist solely in the capacity of the isomorphism $F_{2d} \simeq \Gamma_{2d}\backslash\mathbb{D}_{2d}$ with a locally symmetric space. We will drop the index $2d$, writing simply $\mathbb{D} = \mathbb{D}_{2d}$ and $\Gamma = \Gamma_{2d}$.

The Baily-Borel Theorem 0.14 includes, as part of its package, an explicit projective compactification

$$\Gamma\backslash\mathbb{D} \hookrightarrow \overline{\Gamma\backslash\mathbb{D}}^{\mathrm{BB}}$$

whose boundary can be roughly described as follows: the boundary is a finite union of modular curves and points. The modular curves and points are in bijection with the $\mathbb{Q}$-parabolic subgroups of $\Gamma_{\mathbb{Q}}$ up to Γ-conjugacy, which are in turn in bijection with the rational isotropic subspaces in $v^\perp \otimes \mathbb{Q}$, up to Γ-action.

Note that since $v^\perp$ has signature $(2, 19)$, any isotropic subspace must have dimension 1 or 2. Let us denote a rank 1 isotropic subspace by $I \subset v^\perp$ and a rank 2 isotropic subspace by $J \subset v^\perp$.

Associated to an isotropic subspace I is a filtration $I \subset I^\perp \subset v^\perp$ (and similarly for J) which we will call the *weight filtration*. The corresponding parabolic subgroups of Γ are simply $\mathrm{Stab}_\Gamma(I)$, $\mathrm{Stab}_\Gamma(J)$. If one wishes to think in terms of matrices, these are exactly the matrices which can be written in block form

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

where the three blocks are, respectively, lifts of I, $I^\perp/I$, $v^\perp/I^\perp$ i.e. of the associated graded of the weight filtration (and same for J).

In the rank 2 case, we have a homomorphism $\mathrm{Stab}_\Gamma(J) \to SL(J) \simeq SL_2(\mathbb{Z})$ and the corresponding boundary stratum

$$B_J \subset \overline{\Gamma\backslash\mathbb{D}}^{\mathrm{BB}}$$

is exactly the upper half-plane modulo the image of this homomorphism. The boundary strata B_I in the rank 1 case are points, and form cusps of the modular curves B_J whenever $I \subset J$.

We have the following fact, proven using Hodge theory and Schmid's 1973 nilpotent orbit theorem:

Theorem 2.11. *Let $(X^*, L^*) \to C^*$ be a degeneration of polarized K3 surfaces. The corresponding period map $C^* \to \Gamma\backslash\mathbb{D}$, limits, in the Baily-Borel compactification, to:*

(Type I) *a point in the interior* $\Gamma\backslash\mathbb{D}$*,*
(Type II) *a point in a modular curve* B_J *with* $\operatorname{rk} J = 2$*,*
(Type III) *a point* B_I *with* $\operatorname{rk} I = 1$*.*

Furthermore, in Type II, the j-invariant of the point of the modular curve agrees with the j-invariant of the double curve E of the Type II Kulikov surface X_0.

This theorem is the first bridge between the geometry of the central fiber X_0 and the limit point in a Hodge-theoretic compactification. But the Baily-Borel compactification is highly singular, and its boundary has a gigantic codimension of 18! While it remembers some small amount of geometric data, it is a very "lossy" compactification.

Theorem 2.12 (Ash-Mumford-Rapaport-Tai 1975)**.** *There are compactifications*

$$\Gamma\backslash\mathbb{D} \hookrightarrow \overline{\Gamma\backslash\mathbb{D}}^{\mathfrak{F}}$$

fibering over the Baily-Borel compactification, depending on certain combinatorial data $\mathfrak{F}$, called a fan. *These can be chosen smooth with simple normal crossing boundary (in the orbifold sense, because of issues concerning finite stabilizers). More generally, they have toroidal singularities and completely explicit boundary stratification.*

These are exactly the type of compactifications we want for F_{2d} as we will be able to understand the geometry at the boundary.

Definition 2.13. A *fan* $\mathfrak{F}$ (for $\Gamma\backslash\mathbb{D}$) consists of the following data: For each Γ-orbit of isotropic line $I \subset v^\perp$, we require a rational polyhedral decomposition $\mathfrak{F}_I$ of the positive cone of $I^\perp/I \otimes \mathbb{R}$ which has finitely many orbits of cones under $\Gamma_I := \operatorname{Stab}_\Gamma(I)$.

Recall that in a quadratic space $\mathbb{R}^{1,n}$ of hyperbolic signature $(1, n)$, the positive cone is one of the two connected components of the positive norm vectors. Note that the signature of $I^\perp/I$ is $(1, 18)$ because $v^\perp$ had signature $(2, 19)$.

The toroidal compactification has the following boundary strata: They are in bijection with the orbits of polyhedral cones in $\mathfrak{F}_I$ under Γ_I. Furthermore, if a cone $\sigma \in \mathfrak{F}_I$ has codimension k, then the corresponding boundary stratum is a finite quotient of $(\mathbb{C}^*)^k$ and one stratum is incident upon another (contains it in its Zariski closure) if and only if the corresponding cones are.

Remark 2.14. There is one exception to the above prescription: When the cone σ is a rational ray on the light cone (i.e. boundary of the positive cone) of $I^\perp/I \otimes \mathbb{R}$.

Such cones lie in an isotropic ray $\overline{J} \subset I^\perp/I$ whose inverse image in $I^\perp$ is an isotropic plane $J \supset I$. Then the corresponding boundary divisor of the toroidal compactification is a finite, fiber-preserving quotient of $J^\perp/J \otimes \mathcal{E}$ where $\mathcal{E}$ is the universal elliptic curve over the corresponding modular curve B_J.

Finally, there are the *semitoroidal compactifications* due to Looijenga. These are somewhat technical to define, so we will not get into the exact details. We record their main properties:

(1) They are a common generalization of the Baily-Borel and toroidal compactifications, but include more.
(2) They also depend only combinatorial information, similar to a fan $\mathfrak{F}$, but one allows cones to be infinitely generated.
(3) Their strata are also finite quotients of tori, except in Type II where they are finite quotients of $L \otimes \mathcal{E}$ for some lattice L.
(4) (Alexeev-Engel 2023) They are exactly the normal compactifications of $\Gamma\backslash\mathbb{D}$ which are dominated by some toroidal compactification, and dominate the Baily-Borel compactification.

3. Recognizable divisors

3.1. **Main theorem.** We have compactifications of F_{2d} constructed by completely different methods, and which have the complementary qualities we want:

The toroidal compactifications $\overline{F}_{2d}^{\mathfrak{F}}$ are mildly singular, with easily understood boundary and stratification. But there are infinitely many, with no one distinguished.

The slc stable pair compactifications $\overline{F}_{2d}^{R}$ associated to a canonical choice of polarizing divisor R naturally have a universal family over them and parameterize some generalization of K3 surfaces. But it is unclear how to describe their boundary. In theory, could have badly behaved singularities. It is not even clear if they contain F_{2d}.

Question 3.1. Are there any slc compactifications associated to a divisor R which are toroidal for some fan $\mathfrak{F}_R$? If so, can we explicitly determine the resulting fan $\mathfrak{F}_R$ from R?

To avoid dragging out the punchline, the answer is “yes" so long as R satisfies the following key condition:

Definition 3.2. Let R be a choice of polarizing divisor for F_{2d}. Then we say that R is *recognizable* if any Kulikov surface X_0 contains a divisor R_0 which is the flat limit of R **along any smoothing** $X \to (C, 0)$.

The critical words here are those in bold. Given a single smoothing of X_0 to a Kulikov model $X \to (C, 0)$, we can always simply take the Zariski closure of $R_t \subset X_t$ for $t \neq 0$, and intersect with the central fiber to get the flat limit R_0. But recognizability additionally asserts that the resulting curve $R_0 \subset X_0$ is *independent of how we smooth* X_0.

Theorem 3.3 (Alexeev-Engel 2023)**.** *If R is a recognizable divisor for F_{2d} then the normalization*

$$(\overline{F}_{2d}^{R})^{\nu} = \overline{F}_{2d}^{\mathfrak{F}_R}$$

is a semitoroidal compactification for some unique semifan $\mathfrak{F}_R$.

Furthermore, the “rational curve divisor” $R = \sum_{i=1}^{n_d} R_i$ is recognizable for all degrees $2d$.

3.2. **Example of degree 2 K3s.** For degree 2 K3 surfaces $X \to \mathbb{P}^2$, the ramification divisor $R \subset X$ is recognizable. Let us now describe the corresponding fan $\mathfrak{F}_R$. There is, up to the Γ-action a unique isotropic line $I \subset v^{\perp}$ and the relevant signature $(1, 18)$ lattice is

$$I^{\perp}/I = \langle -2 \rangle \oplus H \oplus E_8^{\oplus 2}.$$

There is a famous list due to Vinberg of hyperbolic lattices whose reflection subgroup is finite index in their isometry group (such a lattice is *reflective*). It turns out that $I^\perp/I$ is in this list and the corresponding Coxeter/Vinberg/Dynkin root diagram is shown in Figure 3.

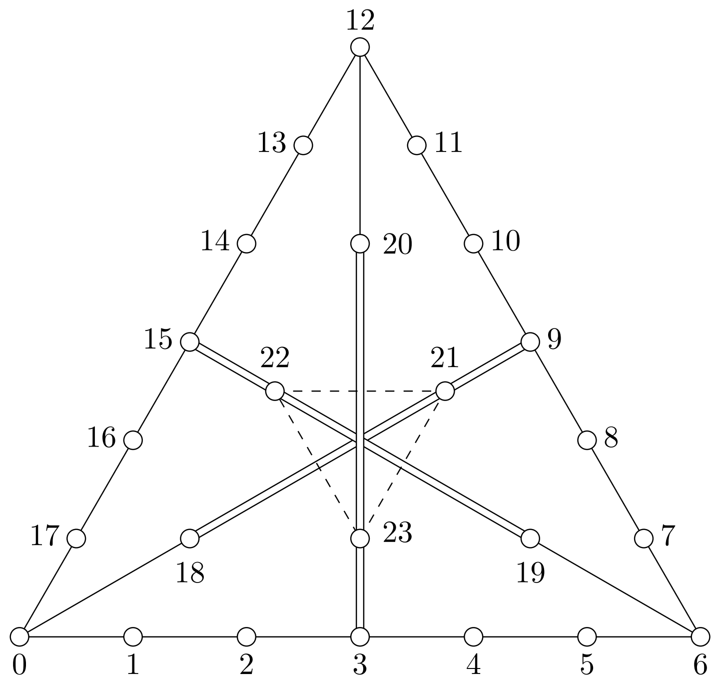

FIGURE 3. The root diagram for $I^\perp/I$.

This says is that $I^\perp/I$ is generated by 24 vectors β_i of norm -2, intersecting according to the graph depicted, and

$$\mathfrak{K} := \{x \in I^\perp/I \otimes \mathbb{R} \,\big|\, x \cdot \beta_i \geq 0\}$$

is a fundamental domain in the positive cone for the finite index subgroup of $O(I^\perp/I)$ generated by reflections in the roots β_i acting by

$$r_{\beta_i}(x) = x + (x \cdot \beta_i)\beta_i.$$

We can construct a fan $\mathfrak{F}_{\rm cox}$ by taking the maximal cones to be $\Gamma_I \cdot \mathfrak{K}$. This "Coxeter fan" is actually *not* the fan we want to take. Rather, we define a new chamber

$$\mathfrak{L} = \Gamma_{\rm irr} \cdot \mathfrak{K}$$

where $\Gamma_{\rm irr}$ is the subgroup generated by reflections in the roots β_{18}, β_{19}, β_{20}, β_{21}, β_{22}, β_{23}. As $\Gamma_{\rm irr}$ is infinite, $\mathfrak{L}$ is infinitely walled. Then we define the maximal cones of $\mathfrak{F}_R$ to be $\Gamma_I/\Gamma_{\rm irr} \cdot \mathfrak{L}$, that is the maximal cones are the orbits of $\mathfrak{L}$. Thus $\mathfrak{F}_R$ is strictly a semifan, as it has cones with infinite numbers of walls. We have:

Theorem 3.4 (Alexeev-Engel-Thompson, 2023)**.** $(\overline{F}_2^R)^\nu = \overline{F}_2^{\mathfrak{F}_R}$.

Furthermore, the normalization ν is necessary, since in $\overline{F}_2^R$ there are different toroidal strata which actually get glued together.

Proof. The lattice points $\lambda \in I^\perp/I$ lying in the positive cone are in bijection with the possible "Picard-Lefschetz transformations" of a degeneration $X^* \to C^*$ in the following sense. Denote the monodromy-transformation as

$$T\colon H^2(X_t) \to H^2(X_t).$$

Then it is unipotent, and has a nilpotent logarithm $N := \log T$. If the limit of the period map is the Baily-Borel boundary point B_I, then N can be written

$$N(x) = (x\cdot\lambda)\delta - (x\cdot\delta)\lambda$$

where $I = \mathbb{Z}\delta$ and $\lambda \in I^\perp/I$ is some vector in the positive cone.

The proof proceeds by "recipe": The input to the recipe is some degeneration $X^* \to C^*$ and the output is the stable model $(\overline{X}_0, \epsilon\overline{R}_0)$.

(1st step) Extract the vector λ encoding the Picard-Lefschetz transform of the degeneration $X^* \to C^*$.

(2nd step) Use the vector λ build a polarized integral-affine sphere B, with a weighted balanced graph $R_{\rm IA} \subset B$ inside it. We give more details below.

(3rd step) Triangulate B and interpret it as the dual complex $\Gamma(X_0)$ of a Type III Kulikov surface via Theorem 1.19. Build a divisor $R_0 \subset X_0$ where $\mathcal{O}_{X_0}(R_0)$ corresponds to $R_{\rm IA}$ via Theorem 1.23.

(4th step) Prove, using deformation theory (particularly work of Friedman), that (X_0, R_0) can be smoothed into (X^*, R^*).

(5th step) Pass to the stable model $(\overline{X}_0, \epsilon\overline{R}_0)$ by contracting curves perpendicular to R_0.

The upshot of all this hard work is that we have very explicit control of the stable model, *as a function of the Hodge-theoretic data* λ. In particular it follows from the construction that:

Proposition 3.5. *The combinatorial type of the stable limit* $(\overline{X}_0, \epsilon\overline{R}_0)$ *is entirely governed by the combinatorial type of the polarized integral-affine sphere* $(B, R_{\rm IA})$*, and in turn by* $\lambda \in I^\perp/I$*.*

Furthermore, the following defines a semifan:

$$\mathfrak{F}_R := \left\{ \begin{array}{c} \text{loci of } \lambda \in I^\perp/I \text{ on which } (B, R_{\rm IA}) \\ \text{is combinatorially constant} \end{array} \right\}$$

It is this semifan which verifies the theorem. □

The critical step of the proof is really the 2nd: How does one input a vector $\lambda \in I^\perp/I$ and output a combinatorial object such as a polarized integral-affine sphere encoding a divisor model? The answer is as follows: First, after an application of an element of Γ_I we may assume $\lambda \in \mathfrak{K}$ lies in the Coxeter chamber for the reflection group. Thus, we are able to extract 24 non-negative *integers*

$$a_i := \lambda \cdot \beta_i \text{ for } i = 1, \dots, 24.$$

These integers actually satisfy five linear relations, since $\lambda \in I^\perp/I$ is a vector in a 19-dimensional lattice.

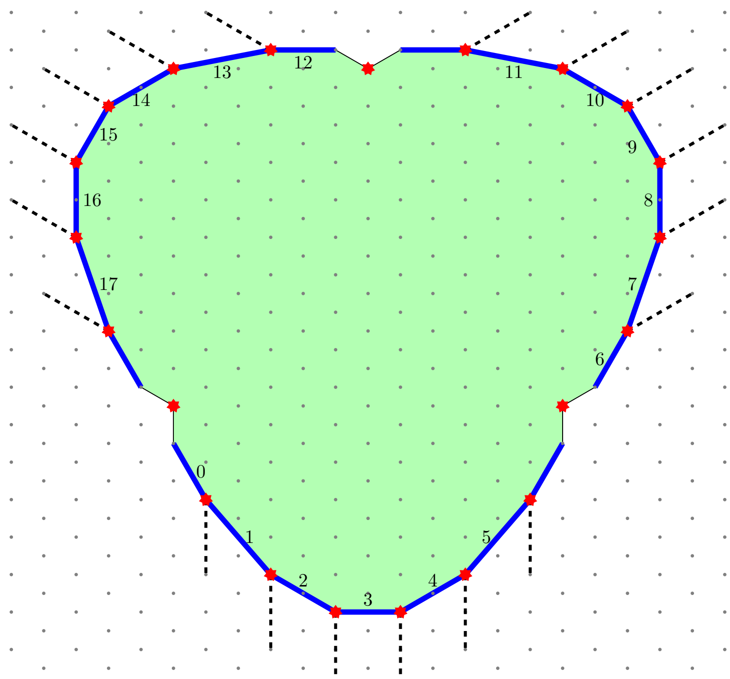

FIGURE 4. Half of a polarized integral-affine sphere associated to the compactificaton of F_2.

From the vector $\vec{a} \in \mathbb{Z}^{24}_{\geq 0}$ we will build a polygon as in Figure 4. First, we successively put vectors $a_i\vec{v}_i$ for $i = 0, \dots, 17$ end-to-end. Here $\vec{v}_i \in \mathbb{Z}^2$ are some fixed primitive integral vectors. Using the values a_i for $i = 18, 19, 20$, we modify this figure by cutting out triangles of affine side lengths a_i from the appropriate sides of the polygon. Finally, we glue two copies of the resulting shape together, sewing up various edges to form an integral-affine sphere B. The "equator" of the sphere, shown

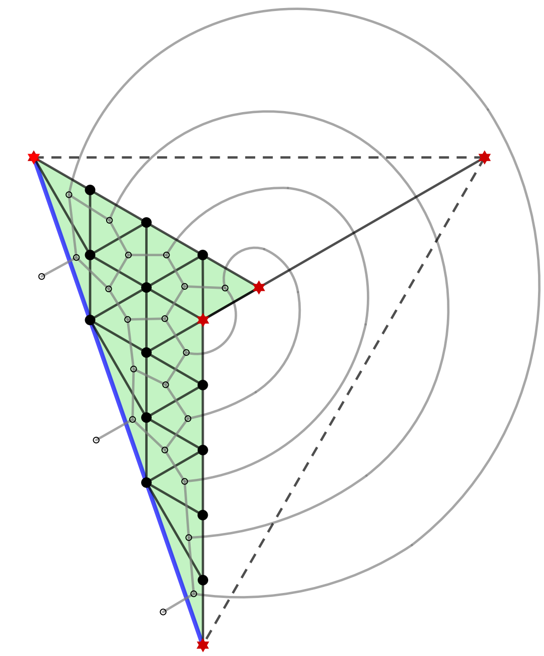

FIGURE 5. Half of an integral-affine sphere, for different parameters $\vec{a}$.

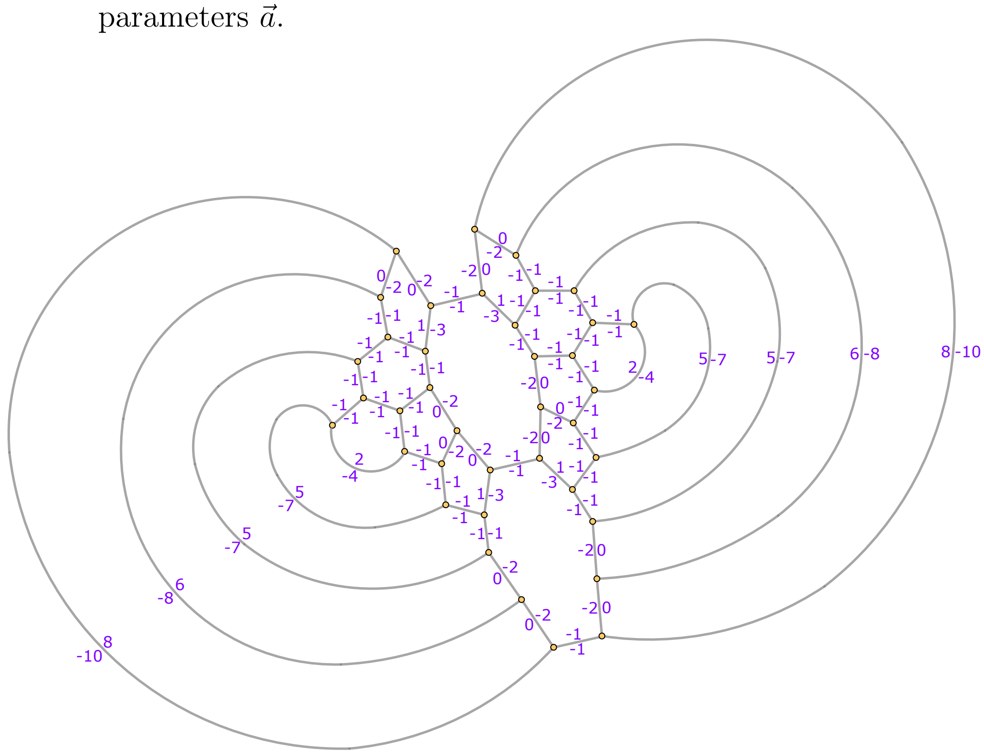

FIGURE 6. The resulting Kulikov surface of Type III.

in blue, becomes $R_{\mathrm{IA}} \subset B$. Another example of the same construction, but for differing value of λ (equivalently $\vec{a}$) is shown in Figures 5, 6.

This procedure may appear unmotivated, but the construction is best understood in the context of mirror symmetry for K3 surfaces.

Due to Dolgachev's 1996 work on lattice-polarized mirror symmetry, the diagram in Figure 3 can be thought of either as:

(A-side) the roots in the lattice $I^{\perp}/I$ where we should put a fan for a toroidal compactification of F_2 or

(B-side) the (-2)-curves on the mirror K3 surface whose perpendiculars bound the ample cone.

Then the construction of the integral affine sphere B in terms of λ can be understood much more easily as a construction in symplectic geometry. One instead constructs a Lagrangian torus fibration of the mirror whose symplectic form has Kähler class corresponding to λ. The key tools in symplectic topology for these constructions come from 2001 work of Symington.

This story generalizes to all moduli of K3 surfaces with involution:

Theorem 3.6 (Alexeev-Engel 2022)**.** *For all moduli spaces of K3 surfaces with involution, there are modular semitoroidal compactifications, associated to the ramification divisor, and one can understand the corresponding semifan explicitly.*

3.3. **Example of elliptic K3 surfaces.** It is also possible to compactify the moduli space of elliptic K3 surfaces using recognizable divisors.

Definition 3.7. An *elliptic K3 surface* is a K3 surface together with a fibration $X \to \mathbb{P}^1$ by genus 1 curves and a section s.

Counted with multiplicity, there are always 24 singular fibers of such a fibration. The subgroup of $\mathrm{Pic}(X)$ generated by s and the fiber f is isometric to the unimodular lattice H. Thus, the elliptic surfaces are "H-polarized K3 surfaces" since they all contain a copy of H in their Picard group. Then are natural choices of polarizing divisor, given by

$$R = s + m \sum_{i=1}^{24} f_i$$

where the sum runs over the singular fibers. Here $m > 0$ is some number. In fact, up to a scaling factor, this divisor agrees with the rational curve divisor for an appropriate choice of m.

In this case, we have a similar result to the degree 2 K3 surfaces:

Theorem 3.8 (Alexeev-Brunyate-Engel 2022)**.** *There is a toroidal compactification of F_{ell} normalizing the stable slc compactification for the* *divisor R. The divisor and stable models admit an explicit description in terms of polarized integral-affine spheres.*

This ends up being nicer than the degree 2 in fact, because we have a fan rather than a semifan. An example of a polarized integral-affine sphere for this case is shown in Figure 7. As usual, it is only possible to draw a fundamental domain for the sphere, together with its integral-affine embedding into $\mathbb{R}^2$.

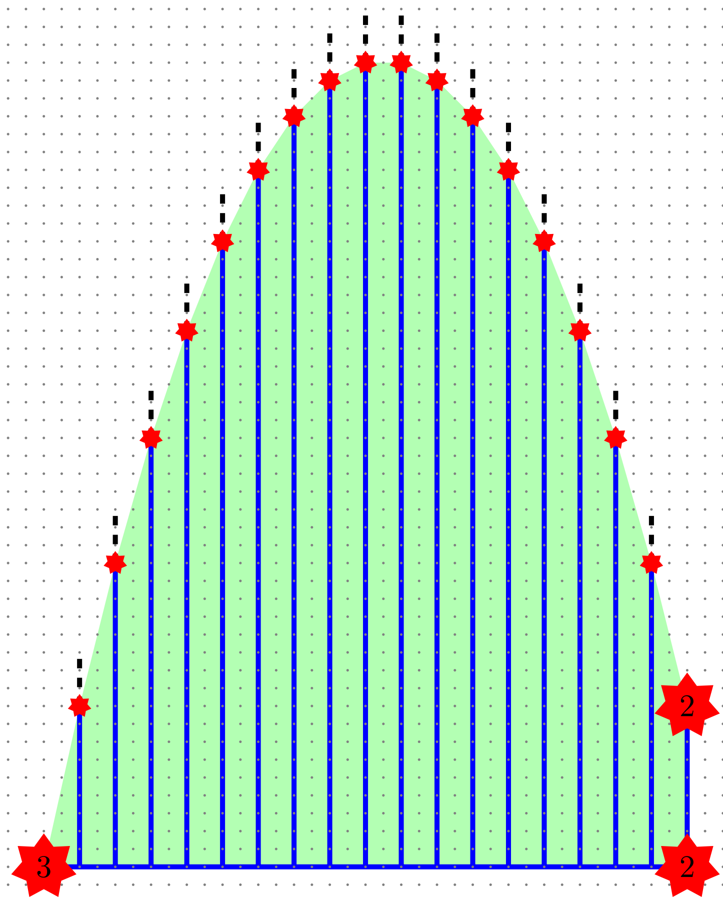

FIGURE 7. Integral-affine sphere for elliptic K3 surfaces.

3.4. **Proof in the general case.** We finally outline the proof Theorem 3.3. The key idea is that the condition of recognizability, Definition 3.2, ensures that the universal pair $(\mathfrak{X}, \mathfrak{R})$ extends over the entire "moduli space of Kulikov surfaces" F_{2d}^{Kul}.

This space is in some ways poorly behaved, but can be intuitively thought of as (some non-separated version of) the "toroidal compactification" associated to a fan $\mathfrak{F}$ consisting of every positive ray in $I^\perp/I$. This space is not finite type, as it contains boundary divisors isomorphic to $(\mathbb{C}^*)^{18}$ for every vector $\lambda \in I^\perp/I$ in the positive cone. But, it does satisfy the valuative criterion for completeness, as any punctured arc in F_{2d} admits a completion in $F_{2d}^{\rm Kul}$ due to the Kulikov-Persson-Pinkham Theorem 1.2.

Laza's Theorem 2.9 proving the existence of a divisor model implies that the open subset $F_{2d}^{{\rm Kul},R} \subset F_{2d}^{\rm Kul}$ on which the universal pair $(\mathfrak{X}, \mathfrak{R})$ defines a divisor model surjects onto the stable slc compactification:

$$F_{2d}^{{\rm Kul},R} \twoheadrightarrow \overline{F}_{2d}^R.$$

Finally, since $F_{2d}^{\rm Kul,R}$ is in some sense toroidal (it is contained in the aforementioned infinite type toroidal compactification), the theorem that the normalization of $\overline{F}_{2d}^R$ is semitoroidal follows from property (4) of semitoroidal compactifications. This proves that recognizable divisors lead to semitoroidal compactifications.

It remains to prove that the rational curve divisor is recognizable. In a degeneration $(X, L) \to (C, 0)$, we may as well base change until the rational curves on the general fiber are not permuted. Then a family of rational curves in $|L^*|$ limits to a so-called *genus* 0 *admissible stable map*—a map from a nodal, arithmetic genus 0 curve, which satisfies a certain matching-tangency condition along all double loci D_{ij}.

One must argue using adjunction that the image of a genus 0 admissible stable map is rigid, perhaps a surprising fact given that X_0 is a union of rational surfaces! This implies the limits of rational curves on X_t to curves on X_0 are actually independent of the smoothing $X \to (C, 0)$. They can be "recognized" from the geometry of X_0 alone, as images of admissible stable genus 0 maps.

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