YRN: COHOMOLOGY OF SHIMURA VARIETIES - PROGRAM

by

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1. Overview

The aim of the workshop is to study the p-adic geometry of Shimura varieties $Sh_K(G)$ via the Hodge–Tate period map $\pi_{HT} : Sh_{K^p}(G) \to \mathscr{F}\ell(G_{\mathbf{Q}_p}, \mu)$, and to prove vanishing results for the cohomology $H^*(Sh_K(G), \mathbf{F}_\ell)$ localised at suitable maximal ideals $\mathfrak{m}$ of the Hecke algebra. We will follow the approach of Caraiani–Scholze [CS17], focusing in particular on how it can be implemented in the setting of Hilbert modular varieties [CT21]. In particular, we will show the following result.

1.0.1. Theorem. — *[CT21, Theorem 7.1.1] Let X be a Hilbert modular variety, $\ell > 2$ and $\mathfrak{m}$ a maximal ideal in the support of $R\Gamma(X, \mathbf{F}_\ell)$ whose associated Galois representation has non-solvable image. Then $R\Gamma(X, \mathbf{F}_\ell)_\mathfrak{m}$ is concentrated in middle degree.*

The proof is based on two key properties of the complex $(R\pi_{HT*}\mathbf{F}_\ell)_\mathfrak{m}$ (in the compact case): (i) it is perverse (up to shift); (ii) its stalks are given by the cohomology of Igusa varieties. Joining these two properties with a comparison theorem for Igusa varieties one can transfer the contribution to $H^*(X, \mathbf{F}_\ell)$ coming from non-ordinary strata to the cohomology (with $\mathbf{Q}_\ell$-coefficients) of a quaternionic Shimura variety attached to a non-split inner form of GL_2. Under a suitable genericity condition on p, such a transfer cannot occur, and the theorem follows.

We will introduce the Hodge–Tate period map, emphasising the role played by local shtukas, and we will study in detail the proof of Theorem 1.0.1, expanding on the above sketch. Most of the arguments should be first discussed for PEL Shimura varieties (excluding type D), as in [CS17], and then adapted to the quaternionic setting. In particular, along the way we will also learn most of the key points in the proof of [CS17, Theorem 1.1]. Finally, we will discuss Koshikawa's alternative approach to vanishing theorems [Kos21], based on vanishing results for the cohomology of local Shimura varieties rather than of Igusa varieties.

2. Talks

2.1. Background on Shimura varieties and Galois representations. — Introduce PEL data and the associated abelian varieties with extra structure [Roz20, §§1, 2]. State the results on integral models in [Roz20, §3]. Then introduce quaternionic Shimura varieties and the associated unitary Shimura varieties [CT21, §3.1] and explain their relation [CT21, Lemma 3.2.7], [CT21, Corollary 3.2.9]. Describe integral models of the relevant unitary Shimura varieties [CT21, §3.3].

Finally, introduce decomposed generic Galois representations [CS17, §6.2] and show that the automorphic representations attached to their lifts cannot be obtained via Jacquet–Langlands transfer (cf. the proofs of [CS17, Lemma 5.4.3] and of [CS19, Corollary 5.1.3]); you may illustrate this in the simplest setting of GL_2, which will be important later (cf. [BH06, Ch. 8, 13]).

Further references. — [Kot92]

2.2. Stratifications on the special fibre of Shimura varieties. — Define isocrystals and the Kottwitz set [Man20, §2.2, 2.3] (see also [Ham15, §3.2, Example 3.3]), and the Newton stratification on the special fibre of PEL Shimura varieties [Man20, §3.2]. Introduce central leaves [Man20, §3.4] and Ekedahl–Oort strata [Ham15, §6.2], and explain their relation [Ham15, Definition 6.6, Definition 6.7]; state [Nie15, Corollary 1.6]. State [Ham15, Corollary 7.8 (2)] on dimension of leaves, and sketch how to deduce it from the corresponding result for minimal Ekedahl–Oort strata [Ham15, §6.1, 6.2]. State [Box15, Theorem C] on affineness of Ekedahl-Oort strata (time permitting, you can comment on the proof).

Finally, state [CT21, Theorem 7.1.1] and explain how to deduce it from [CT21, Proposition 7.2.6] (cf. [CT21, §§7.1, 7.2]). You can admit [CT21, Theorem 2.2.1], [CT21, Lemma 2.3.1], and be brief about the proof of [CT21, Lemma 7.1.8].

Further references. — [VW13], [GK19]

2.3. Igusa varieties. — Define (perfect) Igusa varieties for PEL type Shimura varieties [CS17, Definition 4.3.1] and prove their basic properties [CS17, Proposition 4.3.3, Lemma 4.3.4, Corollary 4.3.5](1). Introduce Mantovan's Igusa tower [CS17, Definition 4.3.6] and explain its relation with perfect Igusa varieties [CS17, Proposition 4.3.8]. Deduce from the results of the previous talk that, for compact Shimura varieties, Mantovan's tower consists of affine schemes, and determine their dimension (cf. [CS19, §3.3](2)).

Finally, introduce the unitary Igusa varieties of [CT21, §4.2] and prove the comparison theorem [CT21, Theorem 4.2.4]; then deduce [CT21, Proposition 7.2.6] from the vanishing statement [CT21, Proposition 7.2.7] (cf. [CT21, §7.2.8]).

Further references. — [Man05]

2.4. The stack of bundles over the Fargues–Fontaine curve and the Newton stratification. — Introduce the adic space "$S \times \mathrm{Spa}\,\mathbf{Z}_p$" [SW20, §11.2] and discuss vector bundles on it [PR22, Proposition 2.1.1, Proposition 2.1.2]. Define the Fargues–Fontaine curve(s) [FS21, Definition II.1.15] (you can restrict to $E = \mathbf{Q}_p$) and the stack Bun_G [FS21, Definition III.1.2]; describe the topological space $|Bun_G|$ [FS21, §III.2]. Define the B_{dR}^{+}-affine Grassmannian [SW20, Definition 19.1.1, Proposition 19.1.2] and Schubert varieties [SW20, §19.2, §19.4] focusing on minuscule ones: in particular, state [SW20, Proposition 19.4.2] (and sketch the proof, time permitting). Define the Beauville–Laszlo map [FS21, p.

(1) Cf. [CT21, Lemma 4.2.2, Lemma 5.1.2] for more detailed proofs of similar statements.

(2) But restrict to the compact case, hence avoid compactifications.

97] and prove [Han21b, Proposition 2.10]. Finally, define the Newton stratification on the flag variety [Han21b, §2.3].

Further references. — [FS22], [CS17]

2.5. Local shtukas and period maps. — Introduce local shtukas with one leg [PR22, Definition 2.2.1] and Breuil–Kisin–Fargues modules [PR22, Definition 2.2.4]; explain how these objects are related [PR22, Definition 2.2.6, Proposition 2.2.7]. Discuss families of local shtukas and their relation with p-divisible groups [PR22, Example 2.3.2]. Then introduce G-shtukas [PR22, Definition 2.4.3] and explain how they give rise to period maps [PR22, §2.5]; in particular, prove [PR22, Proposition 2.5.1] and state its consequences [PR22, Proposition 2.5.2, Proposition 2.5.3]. Finally, define the good reduction locus of a Shimura variety (cf. [CT21, p. 26], [CS19, p. 30]) and use [PR22, Example 2.3.2], [PR22, Proposition 2.5.3] to define the Hodge–Tate period map on the good reduction locus in the PEL case.

Further references. — [CS17], [SW20]

2.6. Rapoport–Zink spaces and the almost product formula. — Define PEL Rapoport–Zink spaces as in [CS17, Definition 4.2.1], and state [CS17, Theorem 4.2.2]. Introduce Rapoport–Zink spaces with infinite level [CS17, Definition 4.2.3] and the local Hodge–Tate period map [CS17, Proposition 4.2.5, Proposition 4.2.6]. Give [CS17, Definition 4.3.11] and prove [CS17, Lemma 4.3.12]. Then prove [CS17, Lemma 4.3.20].(3) Finally, deduce [CS17, Theorem 4.4.4], and state the analogous result in the setting of [CT21], [CT21, Proposition 5.3.1] (you can omit the proof, whose arguments are similar to the previous ones).

Further references. — [Man20]

2.7. Perverse sheaves on flag varieties. — Introduce t-structures and discuss their basic properties [Dim04, pp. 125-127]. Explain how to glue t-structures [Kli, Definition 16.1, Theorem 16.2] (you can be brief about the proof). Then define the perverse t-structure on (a quotient of) the flag variety [Tamb, §0.4] and prove [Tamb, Theorem 0.1.1]. Deduce [Tamb, Corollary 0.5.5]. Time permitting, you may mention the connection with semismall maps [Max19, pp. 156-160].

Further references. — [dCM09], [BBDG18]

2.8. Structure of the ordinary locus. — Define Scholze's moduli spaces of local shtukas [SW20, §23.1] and discuss the case with zero legs [SW20, §23.2] and one leg [SW20, §23.3]. State [SW20, Corollary 24.3.5] comparing Rapoport–Zink spaces and moduli spaces of local shtukas (time permitting, you can sketch the proof [SW20, Lecture 25]). Describe the μ-ordinary Rapoport–Zink spaces in [CT21, §6.2] and the μ-ordinary locus in the corresponding Shimura varieties at infinite level [CT21, Theorem 6.3.3]. Time permitting, you may discuss some of the results of [GI16] used in [CT21, §6].

Further references. — [Han21a]

2.9. Vanishing theorems for the cohomology of Shimura varieties. — Prove [CS17, Theorem 6.3.1](4), admitting [CS17, Corollary 5.5.5, Theorem 5.5.7] (time permitting, you can say something about the proof of these results). Then prove [CT21, Proposition 7.2.7] - cf. [CT21, §7.4] - hence finishing the proof of [CT21, Theorem 7.1.1].

(3) You can find more details in [CT21, §5.1, 5.2], but you should discuss these in the technically simpler setting of [CS17].

(4) Use the results in talk 7 rather than Corollary 6.1.4 in *loc. cit.*

2.10. Vanishing of the cohomology of local Shimura varieties. — The aim of this talk is to explain the proof of [Kos21, Theorem 1.1]. Introduce the space of cocycles $Z^1(W_E, \hat{G})$ [FS22, §§5.1.1, 5.1.2] (you can use the more elementary approach of [DHKM20], cf. pp. 4, 5). Introduce the spectral Bernstein centre and the map to the usual Bernstein centre [FS22, §5.3.1]; sketch how this map is constructed from excursion operators [FS22, §5.3.2], and deduce the existence of semisimple Langlands parameters [FS22, Theorem 5.10]. Then prove [Kos21, Lemma 3.1] and [Kos21, Theorem 1.1] (cf. [Kos21, §4]).

Further references. — [FS22], [Sch, Lectures 26, 27]

3. Examples, questions and problems

3.1. Examples and complements. — The following examples, questions and complements may help better understanding the content of the talks, and could be discussed in the afternoon sessions.

1. (related to §2.1) The proof of Theorem 1.0.1 ultimately rests on the description of the cohomology of quaternionic Shimura varieties with characteristic zero coefficients in terms of quaternionic automorphic forms, which one should keep in mind; cf. [Fre90, Chapter 3].
2. (related to §2.1, §2.2) The reason why vanishing results in the style of Theorem 1.0.1 are non-trivial is that the cohomology groups $H^*(X, \mathbf{Z}_\ell)$ may contain torsion; can you produce an example?
3. (related to §2.2) For GL_2 over a totally real field F and p totally split in F, one only has two strata for each p-adic place, and one does not see the difference between the various stratifications. The situation starts being more interesting in the following examples: (i) p inert in F, cf. [GO00] (ii) Siegel modular varieties, starting from threefolds, cf. [Pri08], [Oor07]. In particular, it is important to understand why there are non-trivial families of supersingular abelian surfaces [KO87].
4. (related to §2.2, §2.3) The key properties of Igusa varieties are established in §2.3 ultimately as a consequence of properties of Ekedahl–Oort strata. Their study makes use of the map $\zeta : Sh_K(G)_{\bar{\mathbf{F}}_p} \to \mathrm{D-Zip}$ [VW13], which can be thought of as a "simple version" of the Hodge–Tate period map. It can be instructive to learn something about the geometry of the map ζ, as well as about the group-theoretic construction of Hasse invariants [GK19]. As a starting point, it is crucial to understand why the μ-ordinary locus is quasi-affine.
5. (related to §2.3) For $GL_2/\mathbf{Q}$, one can prove more directly that the supersingular Igusa variety is described in terms of a definite quaternion algebra, generalising a classical argument of Deuring [Deu41] (and avoiding auxiliary unitary groups). See [How22] (you may learn the argument by rewriting it for Shimura curves over $\mathbf{Q}$).
6. (related to §2.4) Understand the picture for GL_2; [mcg19, Chapter 5] can be helpful. Compare Bun_G to the stack of vector bundles over a smooth projective curve.
7. (related to §2.5) Compare the results in the talk with the mod p version of the story in [XZ17, §§5, 7].
8. (related to §2.6) The stratifications on (the good reduction locus of) $Sh_{K^p}(G)$ coming from the special fibre and from the flag variety coincide on rank one points, but are not equal. This difference is due to the behaviour of π_{HT} on higher rank points, and is crucial for the strategy of [CS17] to work (make sure you understand why). An example which could help understanding what is going on is given in [Tama].
9. (related to §2.7) (i) Compare π_{HT} for a Shimura curve over $\mathbf{Q}$ to a map of smooth projective surfaces contracting finitely many curves. (ii) For GL_2 over a totally real field F and p totally split in F, one can define a "partial" Hodge–Tate period map for each p-adic place of F. Study its geometry; do the results of the talk extend to this

setting? (iii) To get some geometric intuition about perverse sheaves, the examples in [Wil] can be enlightening.

10. (related to §2.8) Understand the relation between Scholze's moduli spaces of local shtukas with one leg and (quotients of) flag varieties. On another note, it can be helpful to look at [Han21a] and, for GL_2, use it to describe the ordinary locus in modular curves of infinite level.
11. (related to §2.9) One topic we omitted from the workshop is the computation of the cohomology of Igusa varieties via trace formulas. In the case of modular curves, you can learn about this in the (very instructive) note [Shi21].

3.2. Problems. —

3.2.1. *The Hodge–Tate period map for Hilbert modular varieties.* — Let F be a totally real field and $G = \mathrm{Res}_{F/\mathbf{Q}} GL_2$.

1. For a prime p inert in F, study the Igusa varieties over $\bar{\mathbf{F}}_p$ attached to G, and extend the comparison theorem [CT21, Theorem 4.2.4]. To begin with, study the relation between the Newton stratification and the Goren–Oort stratification [GO00]. Then look at Tian–Xiao's work on the latter [TX16]; the arguments in *loc. cit.* should be helpful in formulating and proving the relevant comparison theorem. One may start by looking at the simplest example of Hilbert modular surfaces, and then move to higher dimension (where one should get comparison theorems for Igusa varieties beyond the basic case).
2. Use the previous point to study the structure of $(R\pi_{HT*}\mathbf{F}_\ell)_{\mathfrak{m}}$. New phenomena should appear (already in the case of surfaces) with respect to [CS17], [CT21], showing in particular why the auxiliary prime p in *loc. cit.* is taken to be totally split in F.

3.2.2. *Vanishing theorems.* — There are (at least) three general tools to prove vanishing theorems for the cohomology of Shimura varieties: (i) study the geometry of the special fibre; (ii) use the period map π_{HT}; (iii) study local Shimura varieties. In [CT21] one combines (i), (ii), whereas [Kos21] uses (ii), (iii) and Boyer's work [Boy19] rests on (i). One can look at other ways to employ these tools to prove vanishing theorems. For instance:

1. In Koshikawa's setting, try to use (i) (similarly to [CT21, §7]) instead of (ii), in order to deduce vanishing theorems from [Kos21, Theorem 1.1]. As a starting point, look at the compact case.
2. In the GSp_4 setting, can one combine (iii) and a comparison theorem (to be proved) for the basic Igusa variety?
3. Can one obtain (generalisations of) Boyer's results using (ii)?

3.2.3. *Langlands functoriality.* — A key point in [CT21] is that, in a sense, "suitable fibres of π_{HT} realise Langlands functoriality"; one can try to use this to prove instances of this functoriality geometrically. For p-adic modular forms, this idea is due to Howe, and one may start by understanding parts of his paper [How22].

1. For $GL_2/\mathbf{Q}$, use the structure of the supersingular Igusa variety plus the parabolically induced structure of the ordinary locus to obtain instances of Jacquet–Langlands transfer.
2. Extend the result of [CT21] on the structure of the μ-ordinary locus to GSp_4 and describe the relevant basic Igusa variety. Can one combine these ingredients to establish instances of Langlands functoriality for GSp_4 (generalising [vH21])?

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