Workshop

“Analysis and Geometry on Discrete Spaces ”

October 7 - 11, 2024

organized by
Sergey Bobkov, Polona Durcik, Alexandros Eskenazis, Irina Holmes Fay, Paata Ivanisvili, Dor Minzer, and Alexander Volberg

Abstracts

Florent Baudier (Texas A&M University)

Bi-Lipschitz and coarse embeddings of diamond graphs

Abstract: Since its use in the early 2000s by Brinkman and Charikar regarding a dimension reduction problem, the geometry of finitely-branching diamond graphs has been thoroughly investigated. For instance, in 2009 Johnson and Schechtman gave a new characterization of the class of uniformly convexifiable Banach spaces in terms of the bi-Lipschitz geometry of the 2-branching diamond graphs. This result is part of the Ribe program which asks for purely metric characterizations of local properties of Banach spaces. The speaker and his coauthors obtained an analog result involving a countably branching version of the diamond graphs as part of the Kalton program (a similar program but for asymptotic properties instead). In order to say something interesting about coarse embeddings or finer quantitative problems, one usually seeks metric invariants in the form of Poincaré type inequalities. Eskenazis, Mendel, and Naor following earlier work of Lee and Naor, recently introduced such an invariant that captures the geometry of 2-branching diamonds. In this talk, we will briefly discuss this local invariant and explain how a new non-standard probabilistic approach to the Kalton program allows us to derive asymptotic metric invariants that capture the geometry of the countably branching diamond graphs.

David Beltran (Universitat de València)

From Kakeya–Brascamp–Lieb inequalities to localised variants of multilinear Fourier restriction

Abstract: The multilinear Kakeya (or Kakeya–Loomis–Whitney) inequalities of Bennett, Carbery and Tao have played a central role in many recent developments in Euclidean Harmonic Analysis. In particular, they imply multilinear Fourier restriction estimates for hypersurfaces, which can be seen as their oscillatory counterparts. Recently, Bejenaru introduced some variants of such oscillatory estimates, in which a gain is possible if the functions are Fourier localised to a neighbourhood of a submanifold of the hypersurfaces involved.

In this talk, we present an alternative approach to that of Bejenaru for obtaining such refined estimates which, in particular, connects them to the theory of Kakeya-Brascamp-Lieb inequalities; these are an

extension of the aforementioned Kakeya-Loomis-Whitney inequalities. Furthermore, we will present generalisations of Bejenaru's estimates under finer localisation conditions on the functions.

This is joint work with Jennifer Duncan (ICMAT, Madrid) and Jonathan Hickman (The University of Edinburgh).

Pandelis Dodos (University of Athens)

Random graphs and nonlinear spectral gaps

Abstract: A difficult problem, due to Pisier and Mendel–Naor, asks whether every regular expander graph satisfies a discrete Poincaré inequality for functions taking values in a Banach space with non-trivial cotype.

Although this problem is open, it has positive answer for Banach spaces with an unconditional basis and cotype $q \geq 2$. This follows by combining a transfer argument due to Naor/Ozawa and a nonlinear embedding due to Odell–Schlumprecht; this approach yields an estimate for the Poincaré constant that depends exponentially on q.

We shall introduce a combinatorial property of regular graphs, that we call *long-range expansion*, and we shall discuss the following results.

(1) For any integer $d \geq 10$, a uniformly random d-regular graph satisfies the long-range expansion property with high probability.

(2) Any regular graph with the long-range expansion property satisfies a discrete Poincaré inequality for functions taking values in a Banach space with an unconditional basis and cotype q, with a Poincaré constant that is proportional to q^{10}. (This estimate is nearly optimal.)

This is joint work with Dylan Altschuler, Konstantin Tikhomirov, and Konstantinos Tyros.

Polona Durcik (Chapman University)

Quantitative norm convergence of triple ergodic averages for commuting transformations

Abstract: We discuss a quantitative result on norm convergence of triple ergodic averages with respect to three general commuting transformations. For these averages we prove an r-variation estimate, $r > 4$, in the norm. We approach the problem via real harmonic analysis, using the recently developed techniques for bounding singular Brascamp-Lieb forms. It remains an open problem whether such norm-variation estimates hold for all $r \geq 2$ as in the cases of one or two commuting transformations, or whether such estimates hold for any $r < \infty$ for more than three commuting transformation. This is joint work with Lenka Slavíková and Christoph Thiele.

Michael Dymond (University of Birmingham)

Lipschitz mappings of discrete metric spaces

Abstract: One way of comparing discrete metric spaces M and N is through the class of Lipschitz mappings $M \to N$. By viewing M and N as discrete models of continuous spaces, it is sometimes possible to apply the powerful techniques from the theory of Lipschitz Mappings in Functional Analysis to problems in the discrete setting. In this talk we will discuss various problems lying in the interface of Discrete Metric Geometry and Functional Analysis. For example, we will consider the question of the best Lipschitz constant with which a given set of n grid points in the plane may be mapped onto a discrete square. We further discuss some recent progress on the problem of extending Lipschitz

mappings defined on discrete sets. This is based on joint work with Eva Kopecká and Vojtěch Kaluža.

Yuval Filmus (Technion)

Approximate polymorphisms

Abstract: A function f which satisfies $f(x \oplus y) = f(x) \oplus f(y)$ for all x, y is an XOR of a subset of the coordinates. Moreover, if a function f satisfies this for most inputs, then it is close to an XOR — a classical stability result known in theoretical computer science as linearity testing. We discuss what happens if we replace $\oplus$ with a different operation (not necessarily binary), showing that stability still holds (with a wrinkle). The proof involves Jones' regularity lemma (a regularity lemma for Boolean functions) as well as the It Ain't Over Till It's Over theorem.

Joint work with Gilad Chase, Dor Minzer, Elchanan Mossel, and Nitin Saurabh.

Li Gao (Wuhan University)

Log-Sobolev inequalities for matrix-valued functions

Abstract: Log-Sobolev inequalities (LSIs) are fundamental tools in analysis with implications in various fields such as probability theory, statistical mechanics, and information theory. In this talk, I will explore LSIs for matrix-valued functions, highlighting a notable dichotomy that: while the standard L_2-LSI, which is equivalent to hypercontractivity, fails completely in the matrix-valued setting, the modified log-Sobolev inequality, the L_1 variant, remains valid for matrix-valued functions for many important examples. This result has interesting implications, including Gaussian-type concentration inequalities for random matrices and quantum Markov semigroups.

Tuomas Hytönen (Aalto University)

A geometric dichotomy for the discrete Hilbert transform

Abstract: I will report on joint work with Assaf Naor in which we identify the rate of growth of the norms of finite Hilbert transforms as a new Banach space parameter ranging continuously from 0 for UMD (unconditional martingale differences) spaces to 1 for non-super-reflexive spaces. All intermediate rates are shown to be attained by explicitly constructed spaces that are necessarily super-reflexive without UMD. With this tool, we also obtain precise quantitative information about the amount of deviation of martingale type from Rademacher type in classical examples of Pisier (1975), where only upper and lower estimates were previously available. With the help of the finite Hilbert transforms, we obtain efficient indirect bounds for the martingale type without having to provide any examples of martingales with extremal behaviour.

Guy Kindler (Hebrew University of Jerusalem)

A polynomial Bogolyubov-type result for the special linear group

Abstract: For a subset A of a finite dimensional vector space over a finite field, It is known that $2A - 2A$ must contain a subgroup of density that is quasi-polynomial in the density of A. We show an analogue result for the special linear group over a finite field, except we achieve a polynomial density

for the subgroup. The main tool used is a new hyper-contractive estimate for the special linear group, that has origins in the study of 2-to-2 games.

Vjekoslav Kovač (University of Zagreb)

Sharp estimates for Gowers norms on discrete cubes

Abstract: We study optimal dimensionless estimates between the Gowers U^k norms and the l^p norms of functions supported on the discrete cube $[n]^d$. Simultaneously, we study sharp inequalities for the related Gowers-type generalized additive energies for subsets of the same cube, in terms of their size. Exact sharp exponents are found when $n = 2$ and then asymptotic estimates are studied when either n or k go to infinity. Subtle inequalities for the Shannon entropy appear in the last task. This is joint work with Tonći Crmarić (University of Split).

José Ramón Madrid Padilla (Virginia Tech)

On convolution inequalities and applications

Abstract: In this talk we will discuss a collection of convolution inequalities for real valued functions on the hypercube, motivated by combinatorial applications.

Stefanie Petermichl (Universität Würzburg)

Banach space geometry and the dyadic shift

Abstract: UMD spaces are such Banach spaces X where we have unconditional convergence of martingale difference series. Bourgain and Burkholder each showed one direction of a characterisation for these spaces, namely that the Hilbert Transform acting on X-valued functions be L^2 bounded. In each one of these estimates, the relationship of the norm is quadratic. It has been a famous open question to answer whether these relations can be improved (UMD conjecture states linear relations). Our aim is to reduce this question to a dyadic (discrete) question. We show that a new dyadic Haar shift operator has two sided linear norm relations with the Hilbert transform if these operators have values in a UMD Banach space X. The methods (and the numeric constants) are new, even if the UMD space is $\mathbb{R}$. Joint work with K. Domelevo.

Joris Roos (University of Massachusetts Lowell)

Isoperimetric and Poincaré inequalities on the Hamming cube

Abstract: The talk will be about certain isoperimetric inequalities on the Hamming cube near and at the critical exponent $1/2$ and closely related L^1 Poincaré inequalities. Joint work with P. Durcik and P. Ivanisvili.

Justin Salez (Université Paris Dauphine)

Entropy and curvature of Markov chains on metric spaces

Abstract: This talk is devoted to the celebrated Peres-Tetali conjecture, which asserts that any Markov chain exhibiting contraction in the Wasserstein distance should also exhibit contraction in relative entropy, by the same amount. I will give a brief historical overview of this question, describe our main result, and illustrate it with several applications. This is based on joint works with Pietro Caputo and Florentin Münch.

Rocco Servedio (Columbia University)

Sparsifying suprema of Gaussian processes

Abstract: We show that the supremum of any centered Gaussian process can be approximated to any arbitrary accuracy by a finite dimensional Gaussian process, where the dimension of the approximator is just dependent on the target error. As a corollary, we show that for any norm Φ defined over $\mathbb{R}^n$ and target error ε, there is a norm Ψ such that (i) Ψ is only dependent on $t(\varepsilon) = \exp\exp(poly(1/\varepsilon))$ dimensions and (ii) $\Psi(x)/\Phi(x) \in [1-\varepsilon, 1+\varepsilon]$ with probability $1-\varepsilon$ (when x is sampled from the Gaussian space). We prove a similar-in-spirit result for sparsifying high-dimensional polytopes in Gaussian space, and present applications to computational learning and property testing. Our proof relies on Talagrand's majorizing measures theorem.

Joint work with Anindya De, Shivam Nadimpalli, and Ryan O'Donnell.

Lenka Slavíková (Charles University in Prague)

Dimension-free Sobolev-type embeddings in the Gauss space

Abstract: The celebrated logarithmic Sobolev inequality of Gross asserts that if u is a function on $\mathbb{R}^n$ whose first-order weak derivatives belong to the space $L^2(\mathbb{R}^n, \gamma_n)$, where $d\gamma_n(x) = (2\pi)^{-n/2} e^{-|x|^2/2}\, dx$ stands for the Gauss measure on $\mathbb{R}^n$, then $|u|^2 \log_+ |u|$ is integrable with respect to γ_n, i.e., u belongs to the Orlicz space $L^2 \log L(\mathbb{R}^n, \gamma_n)$. In addition, the corresponding estimate of the $L^2 \log L$-norm of u by its L^2-based Sobolev norm holds with a constant independent of the dimension n.

In this talk, I will discuss extensions of Gross' inequality in which $L^2(\mathbb{R}^n, \gamma_n)$ is replaced by a more general function space, with a particular emphasis on the class of Orlicz spaces. Variants of these results involving higher-order derivatives will be considered as well. I will explain how the presented results relate to the Gaussian isoperimetric inequality. I will also compare the Gaussian Sobolev embeddings to their counterparts on the Euclidean space $\mathbb{R}^n$ equipped with the standard Lebesgue measure.

Błażej Wróbel (Polish Academy of Sciences and University of Wrocław)

Dimension-free estimates for low degree functions on the Hamming cube

Abstract: We discuss dimension-free L^p inequalities, $1 < p < \infty$, for low degree scalar-valued functions on the Hamming cube. More precisely, for any $p > 2$, $\varepsilon > 0$, and $\theta = \theta(\varepsilon, p) \in (0,1)$ satisfying

$$\frac{1}{p} = \frac{\theta}{p+\varepsilon} + \frac{1-\theta}{2}$$

we obtain, for any function $f : \{-1,1\}^n \to \mathbb{C}$ whose spectrum is bounded from above by d, the Bernstein-Markov type inequalities

$$\|\Delta^k f\|_p \le C(p,\varepsilon)^k\, d^k\, \|f\|_2^{1-\theta}\|f\|_{p+\varepsilon}^{\theta}, \qquad k \in \mathbb{N}.$$

Analogous inequalities are also proved for $p \in (1,2)$. As a corollary, if f is Boolean-valued or $f\colon \{-1,1\}^n \to \{-1,0,1\}$, then we obtain the bounds

$$\|\Delta^k f\|_p \le C(p)^k\, d^k\, \|f\|_p, \qquad k \in \mathbb{N}.$$

The constant $C(p,\varepsilon)$ depends only on p and ε; crucially, it is independent of the dimension n. We also provide a counterpart of these results on tail spaces.

The talk is based on joint work with Komla Domelevo, Polona Durcik, Valentia Fragkiadaki, Ohad Klein, Diogo Oliveira e Silva, and Lenka Slavíková. It grew out of the workshop *Analysis on the hypercube with applications to quantum computing* held at the American Institute of Mathematics (AIM) in June 2022.

Quanhua Xu (Université de Franche-Comté)

Vector-valued Littlewood–Paley–Stein theory for semigroups

Abstract: This is a survey talk on the vector-valued Littlewood-Paley-Stein theory that we have developed since 1998. The talk will start with the classical Littlewood-Paley-Stein inequality for symmetric diffusion semigroups, then move to the vector-valued setting. We will fucus on our latest work on the case of semigroups of positive contractions on $L_p(\Omega)$ for a <u>fixed</u> $1 < p < \infty$.

One of our latest results asserts the following:
Let $\{T_t\}_{t>0}$ be a strongly continuous semigroup of positive contraction on $L_p(\Omega)$. If a Banach space X is of martingale cotype q, then there is a constant C such that

$$\left\| \left(\int_0^\infty \|t\frac{\partial}{\partial t} P_t(f)\|_X^q \,\frac{dt}{t} \right)^{\frac{1}{q}} \right\|_{L_p(\Omega)} \le C\,\|f\|_{L_p(\Omega;X)}\,, \quad \forall\, f \in L_p(\Omega;X),$$

where $\{P_t\}_{t>0}$ is the Poisson semigroup subordinated to $\{T_t\}_{t>0}$. The least constant C satisfies

$$C \lesssim \max\big(p^{\frac{1}{q}},\, p'\big)\mathsf{M}_{c,q}(X),$$

where $\mathsf{M}_{c,q}(X)$ is the martingale cotype q constant of X. Moreover, the order $\max\big(p^{\frac{1}{q}},\, p'\big)$ is optimal as $p \to 1$ and $p \to \infty$. If X is of martingale type q, the reverse inequality holds. If additionally $\{T_t\}_{t>0}$ is analytic on $L_p(\Omega;X)$, the semigroup $\{P_t\}_{t>0}$ above can be replaced by $\{T_t\}_{t>0}$ itself.

Our approach is built on holomorphic functional calculus. It yields the optimal orders of growth on p of most of the relevant constants. In particular, we resolve a problem of Naor and Young on the optimal order of the best constant in the above inequality when X is of martingale cotype q and $\{P_t\}_{t>0}$ is the classical Poisson and heat semigroups on $\mathbb{R}^d$.

Haonan Zhang (University of South Carolina)

On the Eldan–Gross Inequality

Abstract: Recently, Eldan and Gross developed a stochastic analysis approach to proving functional inequalities on discrete hypercubes. Motivated by a conjecture of Talagrand, one of their main results

relates to the sensitivity, variance, and influences of Boolean functions. In this talk, I will discuss an alternative proof of this inequality based on hypercontractivity and an isoperimetric-type inequality of the heat semigroup. Our proof also extends to biased hypercubes and continuous spaces with positive Ricci curvature lower bounds in the sense of Bakry and Émery. This is based on joint work with Paata Ivanisvili.
