Report on the Trimester Program

Prospects of Formal Mathematics

May 6 - August 16, 2024

Organizers: Kevin Buzzard, Jacques Carette, Michael Kohlhase, Valeria de Paiva, Josef Urban

Topics

The principal topic, embedded in the title: "Formal Mathematics". This was divided into sub-topics, namely the act of formalizing, bridging informal and formal, and libraries of digitized mathematics. These became the three workshop themes.

Scientifically, the topics covered were formalizations themselves (as being artifacts of study), foundations, proof assistants, artificial intelligence, and machine learning in theorem proving, links with computer algebra (i.e. mathematical computation), natural language processing tools for mathematical text, and mathematical knowledge management.

Goals

The overarching goal was to provide a suitable environment, in both time and space, to have experts exchange knowledge via both formal and informal talks. While various pockets of the people who attended already knew each other, there were many different communities represented that rarely interact. An important goal was to rectify this situation.

A secondary goal was to give the opportunity to various sets of people to *get work done*. We did not presuppose what projects they would work on, just that if they were given enough time to be in the same room, exciting things would happen, as they did.

Organization

The trimester program was organized around four major meetings. There was a 'School on Formal Mathematics' starting the program and 3 Workshops, over the course of the trimester. Namely

1. Formalizations in Mathematics

2. Bridging Between Informal and Formal Mathematics

3. Libraries of Digitized Mathematics

There was an extra meeting, 'Women in Formal Mathematics', over the weekend of July 6th and 7th.

We had promised two talks per week, in a series of internal seminars, but ended up having one talk a day mostly. The full listing of titles and speakers is in the appendix.

Results

There were a total of 109 (!!) talks given (plus 12 more at the extra meeting), as well as 36 *impromptu* workgroups. A lot of code was written in many languages. The workgroups involved many, many discussions, somewhat guided by a leader, but often naturally evolved.

The "results" of these discussions are harder to quantify. But qualitatively, they were a resounding success. All participants uniformly reported on just how much *food for thought* these discussions held. The *intellectual mixing* was also remarked upon as being particularly stimulating.

One example of the scale of collaboration is Mario Carneiro (a postdoc who stayed for 90 days), whose final number of collaborative projects worked on during the trimester reached 40. While this is extreme, it demonstrates a unique (and transformative) aspect of formal math: facilitating collaboration between (possibly many) experts with different background and skills.

It was also clear that quite a few people were working on papers (but we don't really have an exact tally). Some were existing collaborations, but many new ones were formed. We have certainly seen HIM credited in a number of recent papers (and blog posts and other social media posts).

Workshops

Formalization of Mathematics workshop

This workshop ran from June 17th to 21st. Other than a panel discussion on Wednesday afternoon, the format was: talks in the morning (typically on the formalization of modern mathematics) and small working groups in the afternoon, hacking on problems. Many experts in the field, covering several theorem provers, presented their work. For example, Cohen and Gonthier talked about their work formalizing nontrivial mathematics in Rocq, Paulson talked about formalizing modern combinatorics in Isabelle/HOL, van Doorn talked about formalizing harmonic analysis in Lean, and Avigad talked about his blockchain-related formalizations. There were other talks which were not about formalizations of specific mathematics, but about tools related to the formalization of mathematics (as organizers this was not something we had in mind at the start, but in fact these turned out to be very interesting talks). Patrick Massot talked about his work informalizing formalized mathematics, and Wojciech Nawrocki talked about his work on interactive widgets for Lean 4, making formalized mathematics more visual.

One of the working groups became a discussion about how to formalize higher category theory in Lean; Emily Riehl and Mario Carneiro began to collaborate at this time, and their work has now become a fully-fledged, large and ongoing project on formalizing Riehl's work on infinity-cosmoi. In contrast to many other formalization projects, there are some highly nontrivial design decisions which one has to make here; the workshop brought Riehl (a world expert in higher category theory) and Carneiro (a world expert in formalization) together, and it was this synergy that enabled the project to be born.

Another interesting project which began at this workshop was James Davenport's project to formalize computable polynomials in Lean; polynomials in Lean's mathematics library are noncomputable, which is good for proving theorems about them but means that Lean cannot be used as a computer algebra system (CAS) without a lot of further work. This is still an interesting open problem.

Bridging Between Informal and Formal Mathematics Workshop

The Bridging between Informal and Formal workshop took place July 8-12, 2024. The format combined morning talks (introducing work on bridging informal mathematics – natural language, diagrams, LaTeX, etc. – with formal systems) with afternoon workgroups exploring specific challenges and hands-on experimentation.

Speakers touched on many aspects. For example, Aarne Ranta discussed his Informath project, i.e., the informalization aspects of Formal Mathematics, Dennis Müller spoke about Injecting Formal Mathematics into LaTeX, Peter Koepke showed his impressive formalization of perfectoid rings in the experimental Naproche system based on controlled natural language, Moa Johansson talked about neuro-symbolic conjecturing, Frederik Schaefer discussed symbolic natural language understanding, Valeria de Paiva and Lucy Horowitz demoed their work on alignments, Wojciech Nawrocki demoed commutative diagrams in Lean, and Josef Urban and Wenda Li addressed autoformalization.

Afternoons were split into workgroups covering topics such as visualization (e.g. commutative diagrams), knowledge graphs of formalized mathematics, errors in formalization, the role of LLMs/neural architectures in autoformalization, proof portability across systems, and joint "deformalization" / narration of mathematics in both informal and formal form.

One workgroup discussed proof portability and formats across proof assistants and natural language; another investigated how neural-symbolic methods might assist lemma discovery and "co-piloting" in proof assistants. The workshop certainly succeeded in bringing together experts in formal proof systems, natural language and symbolic reasoning, and interface/notation design – generating cross-fertilization of ideas and new questions about how to mediate between the informal mathematical discourse and the rigor of formal proof.

Libraries of Digitized Mathematics Workshop

This workshop ran from July 21 to August 2nd, again following the same format. Here the principal concern was one of *scale*. The participants had all experienced that non-trivial shifts in difficulty occur once a library reaches a certain size. The detailed symptoms varied widely, as did many of the suggested solutions. But all agreed that techniques borne out of software engineering were necessary ingredients, even though the topic (mathematics) did not seem to resemble "software" very much (Curry-Howard correspondance notwithstanding). While most of the concerns here were familiar to computer scientists and software engineers, what came out as different is that the very *structure of mathematics* may well be of tremendous help with respect to scaling issues. Some old mathematics (Universal Algebra) and its modern descendants kept showing up.

Informal workshop on formalising the global Langlands Conjectures

This was a series of lectures by Buzzard around the question: is it even possible to *state* the global 'Langlands conjectures' in a theorem prover? The prover used for the workshop was Lean, because its mathematics library contains many of the basic tools needed as prerequisites (Lie algebras, manifolds, and so on). Although Buzzard made it pretty clear at the start that he thought the answer was going to be "we can get nowhere near this, for lots of reasons", observations of Patrick Massot and Mario Carneiro on the first day made him much more optimistic that it would be possible to get *somewhere*, at least in the special case that the group in question was GL_n. Working with a general connected reductive group was (and remains) hopeless, as we still have no formal definition of a connected reductive group in any system.

The workshop ran as follows: Buzzard gave lectures each morning, and people would hack in the afternoons. The results were surprisingly (at least to Buzzard) positive; whilst we did not state the global Langlands conjectures (unsurprisingly), it turned out that it was possible to give a rigorous definition of an automorphic form for the group GL_n over the rationals, and a full formalisation of this definition now exists in Buzzard's FLT repo, the repository whose main aim is to formalise a proof of Fermat's Last Theorem. The Langlands conjectures are statements about automorphic forms, so this work is a necessary prerequisite.

Two other extremely pleasant outcomes also occurred as a result of this work; one began as a joke, and the other was (to Buzzard, at least) very surprising. The joke outcome: as any computer scientist knows, 0 is a natural number, and hence one could ask what the global Langlands conjectures for the group GL_0 were, and if they were provable. This kind of "emptysetology" is amusing, but actually it is an interesting exercise to figure out what the global Langlands conjectures say for GL_0 (the trivial group), as usually people only consider them for GL_n with $n \geq 1$, the case $n = 1$ being the deep theorems of global class field theory, one of the highlights of 19th century mathematics. Indeed, during the week, it was possible to completely explicitly compute all the automorphic forms and representations for GL_0 (there is one), to decide whether it is cuspidal, to check that it is algebraic (although this was not formalised) and to prove that there is an associated p-adic family of Galois representations (this was). Formalising the concept of a p-adic family of Galois representations led Buzzard to the project of formalising the definition of a Frobenius element, a project which was ultimately completed soon after the HIM meeting.

The more serious outcome of the workshop; as an outgrowth of this Langlands project, the following question was raised. If R is a topological ring acting on a topological abelian group M, we say that M is a *topological R-module* if the action $R \times M \to M$ is continuous. Now say that R is a topological ring acting on an abstract (no topology) abelian group M. Is there some kind of "canonical" or "best" topology that one can put on M in order to make it into a topological R-module? In other words, does M inherit a topology from R somehow, given the action of R on M? This question turned out to be a little subtle, not least because it is not a well-defined mathematical problem. The idea would be to define a topology and then prove that it has many agreeable properties (for example, it would be nice if any R-linear map between two R-modules equipped with this topology was automatically continuous). Buzzard, working together with two Bonn undergraduates (Hannah Scholz and Ludwig Monnerjahn), came up with a candidate definition for this topology, and Scholz and Monnerjahn then formalised many properties of this topology (for example, all linear maps are automatically continuous, the topology behaves well under finite direct sums, and also under tensor products when some finiteness conditions hold). Variants of these results made it into Lean's mathematics library `Mathlib`. Note in particular that, in stark contrast to most of the mathematics in `Mathlib`, this is formalized mathematics for which no paper proof exists; as far as we know, the only informal explanation of the work is in the extensive comments in the Lean files.

For Scholz, this project was a kickstarter for her formalization career; she went on to work with Floris van Doorn on formalising the theory of CW complexes in 2025, and this work has also made it into `Mathlib`. We are rather proud that the workshop was able to inspire Bonn undergraduates into engaging with formalisation of interesting mathematics.

Women in Formal Mathematics Workshop

The *Women in Formal Mathematics 2024* workshop took place on July 6–7, 2024, as part of the Hausdorff Institute of Mathematics trimester program *Prospects of Formal Mathematics*, with partial support from *Women in EuroProofNet (WEPN)*[1].

The workshop aimed to highlight the contributions of women researchers in formal mathematics and automated deduction, while fostering community and collaboration.

The program featured five invited speakers: Prof. Ursula Martin (Oxford), Prof. Sandra Alves (Porto), Prof. Brigitte Pientka (McGill), Dr. María Inés de Frutos-Fernández (then at UAM), and Prof. Mateja Jamnik (Cambridge), as well as some contributed talks. The program is attached. Recordings of several talks are available on the HIM YouTube channel[2].

Alongside the inspiring scientific presentations, a social event provided further opportunities for networking and informal exchange. We thank Prof. Martin for helping support this. We also thank the HIM staff, especially Silke Steinert-Berndt and Stefan Hartmann, for their support in making the workshop possible.

[1] `https://europroofnet.github.io/women-epn-2024/`

[2] `https://www.youtube.com/playlist?list=PLul8LCT3AJqSUY15OHv0qtRYTDxlTVzQC`