HAUSDORFF RESEARCH INSTITUTE FOR MATHEMATICS

UNIVERSITÄT BONN

Introductory School

"Definability, Decidability, and Computability"

September 15 - 19, 2025

organized by
Valentina Harizanov, Philipp Hieronymi, Jennifer Park, Florian Pop, Alexandra Shlapentokh

Abstracts

Sylvy Anscombe (Université Paris Cité)

Sixty years of Ax–Kochen–Ershov transfer principles

Abstract: Sixty years ago Ax, Kochen, and (independently) Ershov showed that the first-order theory of a field complete with respect to a non-Archimedean absolute value of characteristic 0 (e.g. $\mathbb{Q}((t))$ or $\mathbb{C}((t))$) is determined by the theories of its residue field and value group. The method is indirect: First, in order to facilitate the use of model theoretic methods, we allow valuations rather than absolute values – the key difference is that the image of the valuation is not required to be Archimedean. Second, we replace the hypothesis of completeness by considering instead those valued fields that are henselian, i.e. that satisfy the conclusion of Hensel's Lemma.

So, for the class of henselian valued fields of equal characteristic zero, one proves a suitable embedding lemma: any pair of embeddings between residue fields and value groups extends to an embedding of valued fields, subject to a natural saturation hypothesis. This embedding lemma in fact goes through allowing constants from certain common subfields, and it yields right away an AKE principle at the level of existential theories. Thus, by a back-and-forth argument, the same holds for any other classical fragment: existential-universal sentences, etc. Perhaps the "state of the art" setting for such AKE principles is the theory of (separably) tame valued fields: these principles and the underpinning algebraic results are due in the main to Kuhlmann and his collaborators, principally Knaf and Pal.

In this short course of five talks I hope to give a rather uniform presentation of AKE principles, beginning in the classical setting of equal characteristic zero, then extending to include those valued fields that are finitely ramified (in mixed characteristic) and (separably) tame. I will describe three families of extensions and applications. The first is to discuss what is known around AKE principles for certain expansions of the language of valued fields, notably difference fields and differential fields. The second is the analysis of existential theories of henselian valued fields, in equal characteristic and to a lesser extent in mixed characteristic. The third is that we lay the ground for the "Taming Theorem" of Jahnke and Kartas, where they find AKE results for the first time that apply to certain valued fields that admit finite extensions with nontrivial defect.

Franziska Jahnke (University of Münster)

Model Theory and Definability

Abstract: The aim of this course is to give a fast-track introduction to model-theoretic methods and their uses in number theory. In the first two to three sessions, I will introduce first-order logic and a range of model-theoretic tools, focusing in particular on compactness, quantifier elimination and definability. For each of the techniques introduced, I will discuss applications to number theory. Additionally, I will set reading tasks for researchers new to the area, enabling them to follow more model-theoretic talks throughout the programme.

In the second part of the course, I will discuss definability of henselian valuations, both explicitly and implicitly, and discuss questions of uniformity and complexity of such definitions.

Bjorn Poonen (Massachusetts Institute of Technology)

Definability in rings of number-theoretic interest

Abstract: Hilbert's tenth problem asked for an algorithm to decide, given a multivariable polynomial equation, whether it has a solution in integers. After Matiyasevich in 1970 completed the proof that no such algorithm exists, the same question has been asked for solutions in other rings and fields. Much has been learned also about what subsets are first-order definable in rings and fields that number theorists and algebraic geometers care about. I will survey these topics, including recent advances and prospects for future study.

Theodore Slaman (University of California, Berkeley)

Computability and Definability: Theory and Application

Abstract: We will start with an overview of the mathematical study of definability. In the context of sets of integers, we will discuss computability, the Halting Problem and its associated operation the Turing jump, arithmetic and hyperarithmetic hierarchies. In the context of the real numbers, the objects are directly related to topological complexity, such as a function's being continuous or a set's being Borel. We will include proofs of several basic theorems chosen to illustrate the basic methods of the area. In the pure theory of definability, we will discuss Martin's Conjecture, which gives a precise sense in which this analysis of definability is intrinsic and inevitable. We will outline two case studies in which the theory of definability is used to study phenomena whose origins are external to mathematical logic: normality to integer bases and Hausdorff dimension.
