Workshop

"Definability in Number Theory and Arithmetic Geometry"

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organized by
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Abstracts

Margaret Bilu (CNRS / Ecole Polytechnique)

Euler products in arithmetic and geometry

Abstract: In this talk I will report on joint work (in progress) with Ronno Das and Sean Howe, where we propose a suitable axiomatic set-up, based on the theory of pre-lambda-rings, in which an *Euler product* operation may be defined. In the context of the Grothendieck ring of varieties this construction recovers the notion of motivic Euler product, which appears in the expression of the limit densities of many natural sequences of moduli spaces. This point of view sheds a new light on some of the properties (and non-properties) of motivic Euler products and on their connection with classical Euler products.

Raf Cluckers (University of Lille)

Finiteness and Diophantine results in Hensel minimal structures

Abstract: I will discuss recent progress in the study of Hensel minimal fields and present joint work with Forey, Halupczok, Loeser, Vermeulen on analogues of Pila-Wilkie's o-minimal counting result for rational points in the Hensel minimal setting.

Nicolas Daans (KU Leuven)

Existential first-order definitions of valuations in function fields

Abstract: Starting from an arbitrary field K, one may form the rational function field $K(T)$: the field of fractions of polynomials in one variable over K. $K(T)$ naturally carries infinitely many K-trivial discrete valuations. One example of such a valuation is the so-called 'degree valuation', whose valuation ring consists of those rational functions $\frac{f}{g}$ given by polynomials f and g satisfying $\deg(f) \leq \deg(g)$.
In 1978 Denef showed that, if K is the field of rational or real numbers, then the valuation ring of the degree valuation has an existential first-order definition in $K(T)$ in the language of rings. His proof crucially relied on the ordered structure on the base field. In the decades thereafter, a myriad of different types of base fields K have been considered for which existential definability of the degree valuation in $K(T)$ could be proven. Perhaps somewhat curiously, the used constructions and techniques

all depend heavily on arithmetic properties of the base field K. To this day, no unified theory explaining all known examples seems to have been formulated.
In this talk, I will give a proof of a case of Denef's original result, discuss how parts of the method can be generalized, and where obstacles lie for our understanding of a general solution. Insofar time allows, this talk may discuss joint work with Karim Johannes Becher and Philip Dittmann.

Julian Demeio (University of Hannover)

The Grunwald Problem for solvable groups

Abstract: Let K be a number field. The Grunwald problem for a finite group (scheme) G/K asks what is the closure of the image of $H^1(K,G) \to \prod_{v\in M_K} H^1(K_v,G)$. For a general G, there is a Brauer—Manin obstruction to the problem, and this is conjectured to be the only one. In 2017, Harpaz and Wittenberg introduced a technique that managed to give a positive answer (BMO is the only one) for supersolvable groups. I will present a new fibration theorem over quasi-trivial tori that, combined with the approach of Harpaz and Wittenberg, gives a positive answer for all solvable groups.

Philip Dittmann (University of Manchester)

Hilbert's 10th Problem in power series rings with parameters

Abstract: We usually think of Hilbert's Tenth Problem as much harder over fields "of global type" such as $\mathbb{Q}$ or $\mathbb{F}_p(t)$ than over fields of analytic nature such as $\mathbb{R}$ or $\mathbb{Q}_l$. However, subtle issues arise in positive characteristic, even for local fields like the formal Laurent series field $\mathbb{F}_p((t))$. Notable progress was made here in particular by Anscombe–Fehm, who showed that Hilbert's Tenth Problem without parameters is decidable over $\mathbb{F}_p((t))$, and Denef–Schoutens, who showed that Hilbert's Tenth Problem with the parameter t is decidable over $\mathbb{F}_p((t))$ assuming Resolution of Singularities, an unsolved conjecture in algebraic geometry. The latter result was later improved in joint work of mine with Anscombe–Fehm.
I will report on ongoing work in this direction, focussing on Hilbert's Tenth Problem in power series fields (or more generally henselian valued fields in equal positive characteristic) with parameters from a trivially valued base field. As an application, it is possible to find wide classes of function fields F such that it is decidable which polynomial equations over F have solutions in almost all completions of F, as well as stronger results under a Resolution of Singularities assumption. Some of this was first explored in joint work with Fehm.

Arix Eggink (Utrecht University)

Diophantine Maps and Hilbert's Tenth Problem for some Noncommutative rings

Abstract: In the first part of my talk, I will introduce Diophantine (equivalence) maps. They are a number theory friendly way to formalize the reduction of Hilbert's tenth problem from one ring (or structure) to another. I will connect Diophantine equivalence maps with interpretations. For the second part I will switch to noncommutative rings. I will define the twisted polynomial ring and some of its variants, such as its left division ring of fractions, and give and partly prove some Hilbert's ten results for them. If time permits, I will also show some results for rings of differential polynomials.

Arno Fehm (TU Dresden)

Linear equations with valuation constraints

Abstract: I will discuss computational problems regarding solvability of linear equations over a global field with constraints on finitely many places. Such problems are always decidable, but their computational complexity depends on the type of constraints and the places used. For example, I will explain why the problem of deciding whether a system of linear equations over the rational numbers has a solution with prescribed p-adic valuation is NP-complete for odd p, but is in P for $p = 2$. Joint work with Manuel Bodirsky.

Philipp Hieronymi (University of Bonn)

A strong version of Cobham's theorem

Abstract: Abstract: Let $k, \ell > 1$ be two multiplicatively independent integers. A subset X of $\mathbb{N}^n$ is k-recognizable if the set of k-ary representations of X is recognized by some finite automaton. Cobham's famous theorem states that a subset of the natural numbers is both k-recognizable and ℓ-recognizable if and only if it is Presburger-definable (or equivalently: semilinear). We show the following strengthening. Let X be k-recognizable, let Y be ℓ-recognizable such that both X and Y are not Presburger-definable. Then the first-order theory of $(\mathbb{N}, +, X, Y)$ is undecidable. This is in contrast to a well-known theorem of Büchi that the first-order theory of $(\mathbb{N}, +, X)$ is decidable. Our work strengthens and depends on earlier work of Villemaire and Bès. The essence of Cobham's theorem is that recognizability depends strongly on the choice of the base k. Our results strengthens this: two non-Presburger definable sets that are recognizable in multiplicatively independent bases, are not only distinct, but together computationally intractable over Presburger arithmetic. This is joint work with Chris Schulz.

Konstantinos Kartas (University of Münster)

Nonstandard methods in almost mathematics

Abstract: Almost mathematics–introduced by Faltings and developed by Gabber–Ramero–is a technical toolkit in commutative algebra and plays a central role in modern p-adic geometry. We present a nonstandard approach based on ultraproducts–originating with Gabber and independently rediscovered by Jahnke and the speaker–that turns "almost" properties into genuine ones. We also present a nonstandard version of the almost purity theorem. Joint work in progress with Franziska Jahnke.

Jochen Koenigsmann (University of Oxford)

Fields with the absolute Galois group of $\mathbb{Q}$

Abstract: We will explain why a field whose absolute Galois group is isomorphic to that of $\mathbb{Q}$ shares many arithmetic properties with $\mathbb{Q}$. This has consequences for the Birational Section Conjecture in Grothendieck's Anabelian Geometry, and it may also lead to new techniques for approaching Hilbert's tenth Problem over $\mathbb{Q}$.

Franz-Viktor Kuhlmann (University of Szczecin)

On certain definable coarsenings of valuation rings and their applications

Abstract: This research [4] originates from the investigation in [1] and [2] of Artin-Schreier and Kummer extensions $(L|K, v)$ of valued fields and the computation of their Kähler differentials and the associated annihilators. In the case of extensions with independent defect (see [1]), a corresponding coarsening of the valuation ring of L is definable in a language of valued rings expanded by a predicate for membership in K. I will describe the applications of this valuation ring and its maximal ideal. In particular, the latter is equal to the unique ramification ideal of the extension. In [3], independent defect is used to define a corresponding valuation ring on K, and if (K, v) is henselian it is shown that it is already definable in the ring language. However, it is the valuation ring on L that we need in our applications (without the assumption of henselianity).
The second, particularly interesting case is that of extensions $(L|K, v)$ of prime degree with $[L : K] = (vL : vK)$. In [2], a presentation of the valuation ring of L as a union over a chain of simple ring extensions of the lower valuation ring is given. From it, a corresponding valuation ring on L is derived that is definable in the above mentioned expanded language. For Artin-Schreier and Kummer extensions, its maximal ideal plays an important role in the presentaion of the Kähler differentials and their annihilators. Also the ramification ideal, this time principal and not equal to the maximal ideal, is definable in the same language.
Finally, constructions will be described of deeply ramified fields in which arbitrarily assigned convex subgroups of the value group are associated with extensions with independent defect.
[1] Steven Dale Cutkosky, Franz-Viktor Kuhlmann, Anna Rzepka: On the computation of Kähler differentials and characterizations of Galois extensions with independent defect, Mathematische Nachrichten 298 (2025), 1549-1577
[2] Steven Dale Cutkosky, Franz-Viktor Kuhlmann: Kähler differentials of extensions of valuation rings and deeply ramified fields, submitted; https://arxiv.org/abs/2306.04967
[3] Margarete Ketelsen, Simone Ramello, Piotr Szewczyk: Definable henselian valuations in positive residue characteristic, J. Symb. Logic, DOI: https://doi.org/10.1017/jsl.2024.55
[4] Franz-Viktor Kuhlmann: On certain definable coarsenings of valuation rings and their applications, in preparation

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Vincenzo Mantova (University of Leeds)

Updates on the exponential-algebraic closedness

Abstract: Zilber's exponential-algebraic closedness conjecture is a form of existential closedness, and it predicts that the graph of exp intersects algebraic varieties as often as feasible, that is to say, compatibly with Ax-Schanuel. If true, it implies strong structural results for complex exponentiation, such as quasiminimality and a form of categoricity. The conjecture can actually be extended to other functions satisfying Ax-Schanuel style properties, such as the j-function. I'll summarise the current unconditional results, mostly about the exponential function and abelian exponentials, and what little we know about definability even assuming the conjectures. I'll then focus on the one-variable case and present the techniques we used for the j-function, where we prove for instance that the second derivative of j has infinitely many 'non-trivial' zeroes, a fact that was seemingly not known (joint work with V. Aslanyan, S. Eterović).

Carlo Pagano (Concordia University)

Hilbert tenth problem for finitely generated rings

Abstract: In this talk we will present the details of a technique, introduced in a joint work with Peter Koymans, that produces elliptic curves with positive but prescribed rank. Building on work

of Poonen—Shlapentokh this allowed to resolve a conjecture of Denef—Lipschitz, stating that $\mathbb{Z}$ is a diophantine subset of the ring of integers of any number field, and in particular allowed to settle Hilbert tenth problem in the negative for all finitely generated infinite commutative rings. The desired elliptic curve provides a reduction to the totally real case, previously established by Denef, building on the classical work of Matyiasevich, Robinson, Davis, Putnam. I will also present more recent work, establishing over any number field the existence of elliptic curves of rank precisely equal to 1.

Marta Pieropan (Utrecht University)

Counting Campana points of bounded height

Abstract: Campana points are an arithmetic notion, first studied by Campana and Abramovich, that interpolates between the notions of rational and integral points. Campana points are expected to satisfy suitable analogs of Mordell's conjecture, Lang's conjecture, Vojta's conjecture and Manin's conjecture, and their study introduces new number theoretic challenges of a computational nature. From a number theoretic points of view, Campana points are characterised as m-full solutions of polynomial equations. This talk focuses on results and techniques to count Campana points of bounded height in the framework of the Manin-type conjecture for Campana points, including my current work with Balestrieri, Brandes, Kaesberg, Ortmann, and Winter.

Tom Scanlon (University of California, Berkeley)

The logical complexity of finitely generated algebras

Abstract: In work with Aschenbrenner, Khélif, and Naziazeno (IMRN, 2018), we described the class of sets definable in (infinite) finitely generated commutative rings noting that while the integers are always interpretable, it can happen that the ring is not bi-interpretable with the integers, even with parameters. In follow up work begun at our Summer 2022 MSRI/SLMath program, Naziazeno and I have extended this analysis to the class of finitely generated (infinite dimensional) algebras over the complex numbers. Now, the basic structure consists of the set of finite sequences of complex numbers with operations for extracting the coordinates of the sequences. We show that the basic structure and each finitely generated infinite dimensional commutative complex algebra are mutually interpretable and express conditions for the bi-interpretability. Notably, we identify the (un)definability of the action of the complex numbers as the key factor to a possible distinction.

Ari Shnidman (Temple University)

Rank stability in quadratic extensions and Hilbert's 10th problem

Abstract: I'll go over our recent proof (joint with Alpoge, Bhargava, Ho) that for every quadratic extension of number fields K/F, there exists an abelian variety over F (which can be taken to the Jacobian of a curve of the form $y^2 = x^p + n$) with positive rank and with the same rank over K. Combining with a previous result of Shlapentokh (and building on the work of many others) this implies a negative answer to Hilbert's tenth problem over the ring of integers of any number field (as was also recently established by Koymans-Pagano via a different method). One of course expects the much stronger statement, that such K-rank stability holds for a positive proportion of elliptic curves (say) over F. If there is time, I will explain what we can say about this question.

Floris Vermeulen (University of Münster)

On Serre's thin set conjecture

Abstract: In the eighties, Serre conjectured upper bounds for counting rational points on thin sets in projective space. Thin sets of type I come from subvarieties, and are well-understood via dimension growth results. So the focus is on thin sets of type II, which correspond to images of ramified dominant finite covers of projective space. I will give an overview of counting rational points on thin sets, and discuss recent work on thin sets of type II via the determinant method. This is based on joint work with Tijs Buggenhout, Raf Cluckers and Tim Santens.

Tingxiang Zou (University of Bonn)

Pairs and stable completions of difference varieties

Abstract: In this talk, we will discuss pairs of models of ACFA (the limit theory of algebraically closed fields in positive characteristic equipped with the Frobenius automorphism) and how to use them to study certain space of definable types in valued difference fields in analogy with Hrushovski–Loeser's stable completions of algebraic varieties. This is a very preliminary step towards a theory of stable completions in the setting of valued difference fields, and many questions remain open for future exploration. This is work in progress, joint with Martin Hils, Ehud Hrushovski and Jinhe Ye.
