

An algebraic analysis of existential quantifiers via Lawvere doctrines

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Introduction

One of the most important notions of categorical logic which enables the study of logic from a pure algebraic perspective is that of a **hyperdoctrine**, introduced by F.W. Lawvere.

By employing these categorical tools, we can describe three distinct types of existential quantifiers as algebras for a particular monad T^{\exists} , defined over the category of doctrines:

- ▶ T^{\exists} -algebras are **existential doctrines**;
- ▶ free T^{\exists} -algebras are existential doctrines satisfying the **existence property**;
- ▶ idempotent T^{\exists} -algebras are existential doctrines equipped with **Hilbert's ϵ -operators**.

F.W. Lawvere (1969), *Diagonal arguments and cartesian closed categories*, In *Category Theory, Homology Theory and their Applications*, Vol. 2.

F.W. Lawvere (1969), *Adjointness in foundations*, *Dialectica*, Vol. 23.

Overview

- ▶ Introduce the notion of existential free elements for Lawvere doctrines
- ▶ Characterize the generalized existential completion
- ▶ Present applications to the Dialectica interpretation

D. Trotta (2020), *The Existential Completion*, Theory and Applications of Categories, 2020, Vol. 35, No. 43.

M.E. Maietti D. Trotta (2023), *A characterization of generalized existential completions*, Annals of Pure and Applied Logic, Vol. 174, No. 4.

D. Trotta, M. Spadetto and V. de Paiva (2023), *Dialectica Principles via Gödel Doctrines*, Theoretical Computer Science, Vol. 947.

D. Trotta, M. Spadetto and V. de Paiva (2021), *The Gödel fibration*, Mathematical Foundations of Computer Science (MFCS).

Hyperdoctrines in categorical logic

- ▶ Hyperdoctrines;
- ▶ fibrations;
- ▶ triposes;
- ▶ elementary and existential doctrines;
- ▶ ...

F.W. Lawvere (1969), *Adjointness in foundations*, *Dialectica*

B. Jacobs (1999), *Categorical logic and type theory*, North-Holland

J.M.E. Hyland, P.T. Johnstone, and A.M. Pitts (1980), *Tripos theory*, *Math. Proc. Camb. Phil. Soc.*

M.E. Maietti, G. Rosolini (2012), *Quotient completion for the foundation of constructive mathematics*, *Log. Univer.*

Conjunctive doctrines

Definition

A **conjunctive doctrine** is a functor $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ from the opposite of the category \mathcal{C} to the category of inf-semilattices.

Conjunctive doctrines

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Example

Let \mathcal{C} be a category with finite limits. The functor

$$\text{Sub}_{\mathcal{C}}: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$$

assigns to an object A in \mathcal{C} the poset $\text{Sub}_{\mathcal{C}}(A)$ of subobjects of A in \mathcal{C} and, for an arrow $B \xrightarrow{f} A$ the morphism $\text{Sub}_{\mathcal{C}}(f): \text{Sub}_{\mathcal{C}}(A) \longrightarrow \text{Sub}_{\mathcal{C}}(B)$ is given by pulling a subobject back along f .

Examples

Example

Let \mathcal{C} be a category with finite limits. The conjunctive doctrine of **weak subobjects (or variations)** is given by the functor

$$\Psi_{\mathcal{C}}: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$$

where $\Psi_{\mathcal{C}}(A)$ is the poset reflection of the slice category \mathcal{C}/A .

Example

Let \mathcal{A} be a locale. The **localic conjunctive doctrine** is given by the functor:

$$\mathcal{A}^{(-)}: \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{InfSL}$$

assigning $I \mapsto \mathcal{A}^I$. The partial order is provided by the pointwise partial order on functions $f: I \longrightarrow \mathcal{A}$.

Syntactic doctrines

Let $\mathcal{L}_{=,\exists}$ be the $(\top, \wedge, =, \exists)$ -fragment of first-order Intuitionistic logic (also called regular). We define a conjunctive doctrine:

$$\mathbf{LT}_{=,\exists}: \mathcal{C}_{\mathcal{L}_{=,\exists}}^{\text{op}} \longrightarrow \mathbf{InfSL}$$

where $\mathcal{C}_{\mathcal{L}_{=,\exists}}$ is the category of lists of variables and term substitutions:

- ▶ **objects** are finite lists of variables $\vec{x} := (x_1, \dots, x_n)$;
- ▶ a **morphism** from (x_1, \dots, x_n) into (y_1, \dots, y_m) is a list $[t_1/y_1, \dots, t_m/y_m]$ where the terms t_i are built in $\mathcal{L}_{=,\exists}$ on the variable x_1, \dots, x_n .

The functor $\mathbf{LT}_{=,\exists}: \mathcal{C}_{\mathcal{L}_{=,\exists}}^{\text{op}} \longrightarrow \mathbf{InfSL}$ sends a list (x_1, \dots, x_n) to the partial order $\mathbf{LT}_{=,\exists}(x_1, \dots, x_n)$ of equivalence classes $[\phi]$ of well formed formulas ϕ in the context (x_1, \dots, x_n) where $[\psi] \leq [\phi]$ if $\psi \vdash \phi$ and two formulas are equivalent if they are equiprovable.

Generalized projections

Definition

A class of morphisms Λ of a category \mathcal{C} is said to be a **left class of morphisms** if:

1. given an arrow fh of \mathcal{C} , if $f \in \Lambda$ and $h \in \Lambda$, then we have $fh \in \Lambda$;
2. pullbacks of arrows in Λ exist for every arrow of \mathcal{C} and for every $f \in \Lambda$ and g of \mathcal{C} , for every pullback square

$$\begin{array}{ccc} D & \xrightarrow{f^*g} & A \\ \downarrow g^*f & \lrcorner & \downarrow f \in \Lambda \\ C & \xrightarrow{g} & B \end{array}$$

we have that $g^*f \in \Lambda$;

3. every isomorphism is in Λ .

Λ -existential doctrines

Definition

Let $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ be a conjunctive doctrine and let Λ be a left class of morphisms of \mathcal{C} . P is called a Λ -**existential** if, for any arrow $f: A \longrightarrow B$ of Λ , the functor $P_f: P(B) \longrightarrow P(A)$ has a left adjoint \exists_f , and these satisfy:

(BC) **Beck-Chevalley condition**;

(FR) **Frobenius reciprocity**.

Notation: When Λ is the class of product projections, we will speak about **pure-existential doctrines**, while when Λ is the class of all the morphisms of the base we will speak about **full-existential doctrines**.

Examples

Example

- ▶ $\text{Sub}_{\mathcal{C}}: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ is Mono-existential

$$\exists_f(g) := f \circ g, \text{ with } f: X \hookrightarrow Y \text{ and } g: Z \hookrightarrow X$$

- ▶ $\Psi_{\mathcal{C}}: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ is full-existential

$$\exists_f(g) := f \circ g, \text{ with } f: X \longrightarrow Y \text{ and } g: Z \longrightarrow X$$

- ▶ $\mathcal{A}^{(-)}: \text{Set}^{\text{op}} \longrightarrow \mathbf{InfSL}$ is full-existential

$$\exists_f(g)(y) := \bigvee_{x \in f^{-1}(y)} g(x), \text{ with } f: X \longrightarrow Y \text{ and } g: X \longrightarrow \mathcal{A}$$

- ▶ $\text{LT}_{=, \exists}: \mathcal{C}_{\mathcal{L}_{=, \exists}}^{\text{op}} \longrightarrow \mathbf{InfSL}$ is pure-existential

$$\exists_{\text{pr}}(\alpha(x, y)) := \exists x. \alpha(x, y)$$

Existential-free elements

Definition

Let $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ be an Λ -existential doctrine. An object β of the fibre $P(B)$ is said to be a Λ -**existential splitting** if for every morphism $g: C \longrightarrow B$ in Λ and for every element γ of the fibre $P(C)$, whenever

$$\beta = \exists_g(\gamma)$$

holds then there exists an arrow $h: B \longrightarrow C$ such that $gh = \text{id}$ and

$$\beta = P_h(\gamma).$$

Moreover, an object α of the fibre $P(A)$ is said to be Λ -**existential-free** if, for every morphism $f: B \longrightarrow A$, $P_f(\alpha)$ is a Λ -existential splitting.

Examples of existential-free elements

Example

- ▶ The Mono-existential-free elements of the doctrine $\text{Sub}_{\mathcal{C}}: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ are exactly the identities arrows;
- ▶ the full-existential-free elements of the doctrine $\Psi_{\mathcal{C}}: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ are exactly the identities arrows;
- ▶ the full-existential-free elements of the doctrine $\mathcal{A}^{(-)}: \text{Set}^{\text{op}} \longrightarrow \mathbf{InfSL}$ are exactly the functions $f: X \longrightarrow \mathcal{A}$ such that every element $f(x)$ is supercompact, i.e. if $f(x) \leq \bigvee_{i \in I} b_i$ then $f(x) \leq b_{\bar{i}}$ for some $\bar{i} \in I$;
- ▶ the pure-existential-free elements of the doctrine $\text{LT}_{=, \exists}: \mathcal{C}_{\mathcal{L}=, \exists}^{\text{op}} \longrightarrow \mathbf{InfSL}$ are exactly the formulae which are free from the existential quantifier.

Enough existential free elements

Definition

We say that a Λ -existential doctrine P has **enough- Λ -existential-free objects** if for every object A of \mathcal{C} , any element $\alpha \in P(A)$ is **Λ -covered** by some element $\beta \in P(B)$ for some object B of \mathcal{C} , namely β is a Λ -existential-free element and

$$\alpha = \exists_g(\beta)$$

with $g: B \longrightarrow A$ arrow of Λ .

Example

- ▶ $\text{Sub}_{\mathcal{C}}: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ has enough-Mono-existential-free objects;
- ▶ $\Psi_{\mathcal{C}}: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ has enough-full-existential-free objects;
- ▶ $\text{LT}_{=, \exists}: \mathcal{C}_{\mathcal{L}_{=, \exists}}^{\text{op}} \longrightarrow \mathbf{InfSL}$ has enough-pure-existential-free objects;

Generalized existential completion

Given a conjunctive doctrine $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ and a left class Λ of morphisms of \mathcal{C} , we can construct a new Λ -existential doctrine

$$\text{Ex}^{\Lambda}(P): \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}.$$

This construction is called **generalized existential completion**. As particular cases, we have the following constructions:

- ▶ **full existential completion:** if \mathcal{C} has finite limits, we can construct a full existential doctrine $\text{fEx}(P)$;
- ▶ **pure existential completion:** if \mathcal{C} has finite products, we can construct an existential doctrine $\text{pEx}(P)$.

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Characterization of generalized existential completion

Theorem

Let $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ be a \wedge -existential doctrine. Then the following are equivalent:

- ▶ $P \cong \text{Ex}^{\wedge}(P')$ for some conjunctive doctrine $P': \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$;
- ▶ P satisfies the following points:
 1. P has enough \wedge -existential-free objects;
 2. P satisfies \wedge -RC, i.e. each top element \top_A of a fibre $P(A)$ is a \wedge -existential-free object;
 3. for every \wedge -existential-free object α and β of $P(A)$, $\alpha \wedge \beta$ is a \wedge -existential-free object.

Examples of generalized existential completions

Example

- ▶ The doctrine $\text{Sub}_{\mathcal{C}}: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ is isomorphic to the generalized existential completion $\text{Ex}^{\text{Mono}}(\mathfrak{t})$ of the trivial doctrine $\mathfrak{t}: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$, i.e. $\mathfrak{t}(X) := \{\bullet\}$ for every object X ;
- ▶ $\Psi_{\mathcal{C}}: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ is isomorphic to the full existential completion $\text{fEx}(\mathfrak{t})$ of the trivial doctrine $\mathfrak{t}: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$;
- ▶ $\text{LT}_{=, \exists}: \mathcal{C}_{\mathcal{L}_{=, \exists}}^{\text{op}} \longrightarrow \mathbf{InfSL}$ is isomorphic to the pure existential completion $\text{pEx}(\text{LT}_{=})$ of the syntactic doctrine $\text{LT}_{=}: \mathcal{C}_{\mathcal{L}_{=}}^{\text{op}} \longrightarrow \mathbf{InfSL}$ associated with the Horn fragment of first order Intuitionistic logic.
- ▶ the localic doctrine $\mathcal{A}^{(-)}: \text{Set}^{\text{op}} \longrightarrow \mathbf{InfSL}$ is a full existential completion if and only if the locale \mathcal{A} is supercoherent.

More examples of generalized existential completions

Definition

A pure existential doctrine $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ is said to be **equipped with Hilbert's ϵ -operators** if for every element α in $P(A \times B)$ there exists an arrow $\epsilon_\alpha: A \longrightarrow B$ such that

$$\exists_{\text{pr}_1}(\alpha) = P_{\langle \text{id}_A, \epsilon_\alpha \rangle}(\alpha)$$

holds in $P(A)$, where $\text{pr}_1: A \times B \longrightarrow A$ is the first projection.

Theorem

A pure existential doctrine $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ is equipped with Hilbert's ϵ -operators if and only if $P \cong \text{pEx}(P)$.

More examples of generalized existential completions

Definition

Let \mathbb{A} be a PCA. We define the **realizability doctrine**

$$\mathcal{P}: \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{InfSL}$$

The partial ordered set $(\mathcal{P}(X), \leq)$ is defined as follows: let $P(\mathbb{A})^X$ denote the set of functions from X to the powerset $P(\mathbb{A})$ of \mathbb{A} . Let \leq be the preorder on this set defined as: $\alpha \leq \beta$ if there exists an element $c \in \mathbb{A}$ such that for all $x \in X$ and all $a \in \alpha(x)$ we have that $c \cdot a$ is defined and $c \cdot a \in \beta(x)$. Then $\mathcal{P}(X)$ is defined as the quotient of $P(\mathbb{A})^X$ by the equivalence relation \sim generated by \leq .

Theorem

The realizability doctrine $\mathcal{P}: \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{InfSL}$ is isomorphic to the full existential completion of the doctrine $\mathcal{P}^{\text{sing}}: \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{InfSL}$ of singletons.

Recap: Dialectica interpretation

Gödel's Dialectica Interpretation: an interpretation of intuitionistic arithmetic HA in a quantifier-free theory of functionals of finite type, called system T.

Idea: translate every formula A of HA to $A^D = \exists x \forall y A_D$, where A_D is quantifier-free.

Application: if HA proves A , then system T proves $A_D(t, y)$, where y is a string of variables for functionals of finite type, and t a suitable sequence of terms (not containing y).

Goal: to be as constructive as possible, while being able to interpret all of classical arithmetic.

Recap: Dialectica interpretation

The most complicated clause of the translation is the definition of the translation of the **implication connective** $(\psi \rightarrow \phi)^D$:

$$(\psi \rightarrow \phi)^D = \exists f_0, f_1. \forall u, y. (\psi_D(u, f_1(u, y)) \rightarrow \phi_D(f_0(u), y)).$$

This involves three logical principles: a form of the **Principle of Independence of Premise** (IP), a generalisation of **Markov Principle** (MP), and the **axiom of choice** (AC).

Intuition: given a witness u for the hypothesis ψ_D , there exists a function f_0 assigning a witness $f_0(u)$ of ϕ_D to every witness u of ψ_D . Moreover, this assignment has to be such that from a counterexample y of the conclusion ϕ_D we should be able to find a counterexample $f_1(u, y)$ to the hypothesis ψ_D .

First-order hyperdoctrines

Definition

A **first-order hyperdoctrine** is a contravariant functor:

$$P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Hey}$$

from a category with finite products \mathcal{C} to the category of Heyting algebras **Hey** satisfying:

1. for every projection $\text{pr}_A: A \times B \longrightarrow A$, the morphism $P_{\text{pr}_A}: P(A) \longrightarrow P(A \times B)$ has a left adjoint \exists_{pr_A} and a right adjoint \forall_{pr_A} , satisfying (BC).
2. For every object A of \mathcal{C} there exists a predicate δ_A of $P(A \times A)$ satisfying for every α of $P(A \times A)$ that:

$$\top \leq P_{\Delta_A}(\alpha) \text{ if and only if } \delta_A \leq \alpha$$

where $\Delta_A: A \longrightarrow A \times A$ denotes the diagonal arrow.

Definition

A first-order hyperdoctrine $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Hey}$ is called a **Gödel hyperdoctrine** if:

1. the category \mathcal{C} is cartesian closed;
2. the doctrine P has enough pure-existential-free predicates;
3. the pure-existential-free objects of P are stable under universal quantification, i.e. if $\alpha \in P(A)$ is existential-free, then $\forall_{\text{pr}}(\alpha)$ is pure-existential-free for every projection pr from A ;
4. the sub-doctrine P' of the pure-existential-free predicates of P has enough pure-universal-free predicates.

An element α of a fibre $P(A)$ of a Gödel hyperdoctrine P that is both an existential-free predicate and a universal-free predicate in the sub-doctrine P' of existential-free elements of P is called a **quantifier-free predicate** of P .

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Notation. From now on, we shall employ the logical language provided by the **internal language** of a doctrine and write:

$$a_1 : A_1, \dots, a_n : A_n \mid \phi(a_1, \dots, a_n) \vdash \psi(a_1, \dots, a_n)$$

instead of:

$$\phi \leq \psi$$

in the fibre $P(A_1 \times \dots \times A_n)$. Similarly, we write:

$$a : A \mid \phi(a) \vdash \exists b : B. \psi(a, b) \text{ and } a : A \mid \phi(a) \vdash \forall b : B. \psi(a, b)$$

in place of:

$$\phi \leq \exists_{\text{pr}_A} \psi \text{ and } \phi \leq \forall_{\text{pr}_A} \psi$$

in the fibre $P(A)$. Also, we write $a : A \mid \phi \dashv\vdash \psi$ to abbreviate $a : A \mid \phi \vdash \psi$ and $a : A \mid \psi \vdash \phi$.

Skolemisation in Gödel hyperdoctrines

Theorem

Let $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Hey}$ be a Gödel hyperdoctrine, and let α be an element of $P(I)$. Then there exists a quantifier-free predicate α_D of $P(I \times U \times X)$ such that:

$$i : I \mid \alpha(i) \dashv\vdash \exists u : U. \forall x : X. \alpha_D(i, u, x).$$

Theorem

Every Gödel hyperdoctrine $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Hey}$ validates the **Skolemisation principle**, that is:

$$a_1 : A_1 \mid \forall a_2. \exists b. \alpha(a_1, a_2, b) \dashv\vdash \exists f. \forall a_2. \alpha(a_1, a_2, fa_2)$$

where $f : B^{A_2}$ and fa_2 denote the evaluation of f on a_2 , whenever $\alpha(a_1, a_2, b)$ is a predicate in the context $A_1 \times A_2 \times B$.

Principle of Independence of Premise in Gödel hyperdoctrines

Theorem

Every Gödel hyperdoctrine $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Hey}$ satisfies the **Rule of Independence of Premise**: whenever $\beta \in P(A \times B)$ and $\alpha \in P(A)$ is an existential-free predicate, it is the case that:

$a : A \mid \top \vdash \alpha(a) \rightarrow \exists b. \beta(a, b)$ implies that $a : A \mid \top \vdash \exists b. (\alpha(a) \rightarrow \beta(a, b))$.

Moreover, if existential-free predicates are closed with respect to finite conjunctions, then P satisfies the **Principle of Independence of Premise**:

$a : A \mid \top \vdash (\alpha(a) \rightarrow \exists b. \beta(a, b)) \rightarrow \exists b. (\alpha(a) \rightarrow \beta(a, b))$.

Modified Markov Principle in Gödel hyperdoctrines

Theorem

Every Gödel hyperdoctrine $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Hey}$ satisfies the following **Modified Markov Rule**: whenever $\beta_D \in P(A)$ is a quantifier-free predicate and $\alpha \in P(A \times B)$ is an existential-free predicate, it is the case that:

$a : A \mid \top \vdash (\forall b. \alpha(a, b)) \rightarrow \beta_D(a)$ implies that $a : A \mid \top \vdash \exists b. (\alpha(a, b) \rightarrow \beta_D(a))$.

Moreover, if existential-free predicates are closed with respect to implication, then P satisfies the following **Modified Markov Principle**:

$a : A \mid \top \vdash (\forall b. \alpha(a, b) \rightarrow \beta_D(a)) \rightarrow \exists b. (\alpha(a, b) \rightarrow \beta_D(a))$.

Markov Principle in Gödel hyperdoctrines

Corollary

Every Gödel hyperdoctrine $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Hey}$ such that \perp is a quantifier-free predicate satisfies **Markov Rule**: for every quantifier-free element $\alpha_D \in P(A \times B)$ it is the case that:

$$b : B \mid \top \vdash \neg \forall a. \alpha_D(a, b) \text{ implies that } b : B \mid \top \vdash \exists a. \neg \alpha_D(a, b).$$

Moreover, if existential-free predicates are closed with respect to implication, then P satisfies **Markov Principle**:

$$b : B \mid \top \vdash \neg \forall a. \alpha_D(a, b) \rightarrow \exists a. \neg \alpha_D(a, b).$$

Strengthened Dialectica functionals

Theorem

Let $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Hey}$ be a Gödel hyperdoctrine such that the existential-free predicates are closed with respect to implication and finite conjunctions and falsehood \perp is a quantifier-free predicate. Then for every ψ_D in $P(I \times U \times X)$ and ϕ_D in $P(I \times V \times Y)$ quantifier-free predicates of P we have that the formula:

$$i : I \mid \exists u. \forall x. \psi_D(i, u, x) \rightarrow \exists v. \forall y. \phi_D(i, v, y)$$

is provably equivalent to:

$$i : I \mid \exists f_0, f_1. \forall u, y. (\psi_D(i, u, f_1(i, u, y)) \rightarrow \phi_D(i, f_0(i, u), y)).$$

Related and Future work

- ▶ Characterize exact and regular completions of doctrines (j.w.w. M. Maietti);
- ▶ Abstract the sheafification as tripos-to-topos-adjunction (j.w.w. M. Maietti);
- ▶ Abstract the notion of “supercampact element” to implicative algebras, and characterize their assemblies (j.w.w. S. Maschio);
- ▶ Categorify computational notions: Medvedev, Muchnik and Weihrauch reducibilities (j.w.w. M. Valenti and V. de Paiva);
- ▶ Establish the formal connection with the Cartesian Bicategories of Carboni and Walters (j.w.w. F. Bonchi and A. Di Giorgio).

D. Trotta and S. Maschio (2024), *On categorical structures arising from implicative algebras: from topology to assemblies*, *Annals of Pure and Applied Logic*

M.E. Maietti and D. Trotta (2024), *Quotients, rure existential completions and arithmetic universes*, *Theory and Applications of Categories* (to appear)

D. Trotta, M. Valenti and V. de Paiva (2024), *Categorifying computable reducibilities*, *arXiv*

F. Bonchi, A. Di Giorgio and D. Trotta, *When Lawvere meets Peirce: an equational presentation of boolean hyperdoctrines*, *arXiv*