Book of Abstracts

Special Topic School: Optimality of adaptive finite element methods

September 22 – 26, 2025

Lecture Series

Carsten Carstensen (Humboldt University of Berlin)

Axioms of Adaptivity

Lecture 1:
The adaptive algorithm, Dörfler marking, the set-up of and the axioms of adaptivity, newest-vertex bisection in 2D, the notion of optimal rates. Overview of the proof. Poisson model problem and conforming P1 FEM. Discrete trace inequality, jump control, proof of stability and reduction.

Lecture 2:
Axioms of adaptivity imply convergence. Stability; reduction; discrete reliability; quasi-orthogonality; estimator convergence; (A12), plain and R-linear convergence uniformly on each level.

Lecture 3:
Axioms of adaptivity imply optimal rates. Stevenson's comparison lemma. Proof of optimal convergence rates. Quasiinterpolation. Proof of (A3)-(A4) in model example.

Lars Diening (Bielefeld University)

Meshes in AFEM

In this lecture series we address adaptive meshes as needed in the adaptive finite element method (AFEM). We present several refinement algorithms in 2D, 3D and general dimension. The bisection method of Maubach and Traxler produces meshes with important fine properties. We will explain those and their need for AFEM:

Shape regularity:
Needed for estimates like Poincaré.

Control of conforming closure:
Needed for the optimality of the AFEM loop with Dörfler marking.

Grading estimates:
Needed for the Sobolev stability of the L^2-projection

Limited genetic diversity:
Needed for the optimality of the AFEM loop with maximal marking.

Moreover, we present a novel bisection method (colored bisection) that allows to use any initial mesh.

Christian Kreuzer (TU Dortmund)

On instance optimal adaptive finite element methods

An instance optimal method calculates a solution for a given error tolerance, such that any solution with a comparable error requires comparable complexity. Ideally, this property is independent of the specific problem within the problem class under consideration.

In our case of an adaptive finite element method (AFEM), the considered error is the discretisation error and complexity is measured in terms of the number of mesh elements or equivalently degrees of freedom.

We will first illustrate fundamental problems of the issue in one dimension. The main part of the lecture will then deal with the two-dimensional Poisson problem, where grid refinement and error localisation become significantly more complex. This requires the development of new structures, like family relations between nodes and their limited genetic diversity; see Figure 1. In dimensions greater than two, the issue is still open.

We will focus the presentation on simple (finite-dimensional) data and discuss possible extensions to more general cases.

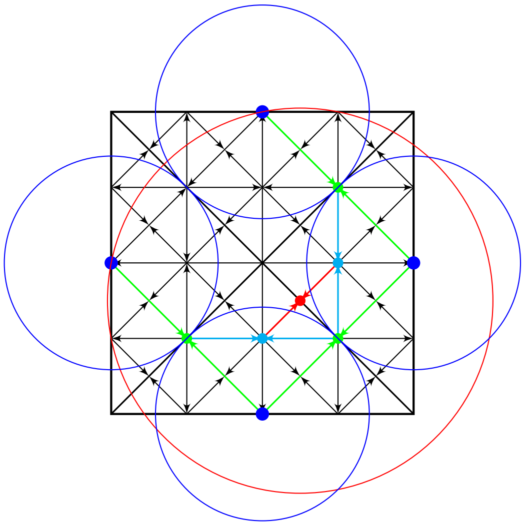

Figure 1: The figure illustrates the proof of limited genetic diversity of nodes.

The lecture is based on

[1] L. Diening, C. Kreuzer, R. Stevenson, *Instance optimality of the adaptive maximum strategy*, Found. Comput. Math., 16(1):33—68, 2016.

Dirk Praetorius (TU Wien)

Adaptive Finite Element Methods: Optimal Rates vs. Optimal Complexity

Following the influential work [2], most works on adaptive finite element methods (AFEMs) aim to prove optimal convergence rates, i.e., optimal decay of the error estimator (or some quasi-error quantity) with respect to the number of degrees of freedom and, equivalently for fixed polynomial degree, with respect to the number of elements. However, at least when nonlinear partial differential equations (PDEs) are concerned, the finite element discretization leads to discrete nonlinear systems that have to be solved by appropriate iterative linearization. Consequently, a practical AFEM must control and balance different error sources, i.e.,

- the discretization error arising from discretizing the PDE by finite elements,
- the linearization error arising from linearizing the nonlinear finite element formulation,
- the algebraic error arising from solving the (practically large) linearized finite element formulation by an inexact iterative solver.

Note that this corresponds to a sequential algorithm with potentially three nested loops. Moreover, any computed finite element approximation to the PDE solution does indeed depend on the full adaptive history. From this perspective, it is clear that optimal convergence rates of AFEM should not only be addressed with respect to the number of degrees of freedom, but rather with respect to the overall computation cost (and hence, with a glance on practice, with respect to the cumulative computational time); see also [1, 3, 7], where this is addressed by a perturbation analysis.

In our presentations, we will address this question by making three steps:

- symmetric linear elliptic PDEs, where the arising linear finite element systems are solved by means of a contractive iterative solver;
- nonsymmetric linear elliptic PDEs, where the arising linear finite element systems are first symmetrized and then solved by means of a contractive iterative solver;
- quasi-linear energy minimization problems with strongly monotone and Lipschitz continuous nonlinearity, where the nonlinear finite element systems are first linearized and then solved by means of a contractive iterative solver.

It will be shown that AFEM with optimal complexity (i.e., guaranteed optimal rates with respect to the overall computation cost) fits nicely in the framework of the *axioms of adaptivity* [4] and requires only additional assumptions on the symmetrization, linearization, and iterative algebraic solver, which can be guaranteed for the considered model problems. Unlike [1, 3, 7], particular focus will be on guaranteed convergence for arbitrary adaptivity parameters [5, 6, 8, 9].

References

[1] R. Stevenson: *Optimality of a standard adaptive finite element method*, Foundations of Computational Mathematics, 7 (2007), 245–269.

[2] J.M. Cascón, C. Kreuzer, R.H. Nochetto, K.G. Siebert: *Quasi-optimal convergence rate for an adaptive finite element method*, SIAM Journal of Numerical Analysis, 46 (2008), 2524–2550.

[3] C. Carstensen, J. Gedicke: *An adaptive finite element eigenvalue solver of asymptotic quasi-optimal computational complexity*, SIAM Journal of Numerical Analysis, 50 (2012), 1029–1057.

[4] C. Carstensen, M. Feischl, M. Page, D. Praetorius: *Axioms of adaptivity*, Computers & Mathematics with Applications, 67 (2014), 1195–1253.

[5] G. Gantner, A. Haberl, D. Praetorius, S. Schimanko: *Rate optimality of adaptive finite element methods with respect to overall computational costs*, Mathematics of Computation, 90 (2021), 2011–2040.

[6] P. Heid, D. Praetorius, T. Wihler: *Energy contraction and optimal convergence of adaptive iterative linearized finite element methods*, Computational Methods in Applied Mathematics, 21 (2021), 407–422.

[7] A. Haberl, D. Praetorius, S. Schimanko, M. Vohralík: *Convergence and quasi-optimal cost of adaptive algorithms for nonlinear operators including iterative linearization and algebraic solver*, Numerische Mathematik, 147 (2021), 679–725.

[8] P. Bringmann, M. Feischl, A. Miraçi, D. Praetorius, J. Streitberger: *On full linear convergence and optimal complexity of adaptive FEM with inexact solver*, Computers & Mathematics with Applications, 180 (2025), 102–129.

[9] A. Miraçi, D. Praetorius, J. Streitberger: *Unconditional full linear convergence and optimal complexity of adaptive iteratively linearized FEM for nonlinear PDEs*, Mathematics of Computation, in print (2025), Preprint on arXiv:2401.17778.

Additional Talks

Ani Miraçi (TU Wien)

Iterative Solvers in Adaptive Finite Element Methods

Numerous physical phenomena are modeled through PDEs[1] and often discretized via FEM[2]. As the unknown true solution may exhibit a singular behavior, AFEMs[3] allow to generate a series of meshes refined towards a singularity by locally refining the local size h of the computational mesh. Furthermore, in many practical problems, employing globally continuous piecewise polynomials of degree $p \geq 1$ to approximate the unknown solution yields improved accuracy and faster convergence.

However, one still needs to design suitable iterative solvers to treat the arising discrete system of linear equations. Indeed, even in the case of linear, symmetric, elliptic PDEs, and assuming a conscientious choice of the discretization parameters h and p has already been made, the remaining linear system may be too computationally demanding to be resolved via a direct solver.

Thus, to make the overall numerical simulation efficient, we design a geometric multigrid method based on [4, 5] and guaranteeing: (1) *linear complexity*, i.e., the computational time is linearly proportional to the size of the problem; (2) *algebraic error contraction* in a *robust* way with respect to the discretization parameters h and p. To show this, we introduce the two central tools for the analysis, namely, a strengthened Cauchy–Schwarz inequality and a robust space stable decomposition employing results from [1] and [2, 3].

References

[1] J. Schöberl, J. M. Melenk, C. Pechstein, and S. Zaglmayr: *Additive Schwarz preconditioning for p-version triangular and tetrahedral finite elements.* IMA J. Numer. Anal., 28(1):1–24, 2008. DOI 10.1093/imanum/drl046

[1] partial differential equations (PDEs)
[2] finite element method (FEM)
[3] adaptive finite element methods (AFEMs)

[2] L. Chen, R. H. Nochetto, and J. Xu: *Optimal multilevel methods for graded bisection grids.* Numer. Math., 120(1):1–34, 2012. DOI 10.1007/s00211-011-0401-4

[3] J. Wu and H. Zheng: *Uniform Convergence of Multigrid Methods for Adaptive Meshes.* Appl. Numer. Math., 113(C):109–123, 2017. DOI 10.1016/j.apnum.2016.11.005

[4] A. Miraçi, J. Papež, M. Vohralík: *A-posteriori-steered p-robust multigrid with optimal step-sizes and adaptive number of smoothing steps*, SIAM J. Sci. Comput. 43, 5, S117–S145, 2021. DOI 10.1137/20M1349503

[5] M. Innerberger, A. Miraçi, D. Praetorius, J. Streitberger: *hp-robust multigrid solver on locally refined meshes for FEM discretizations of symmetric elliptic PDEs*, ESAIM: Mathematical Modelling and Numerical Analysis, 58, 247–272, 2024. DOI 10.1051/m2an/2023104

Ngoc Tien Tran (University of Augsburg)

Convergent adaptive hybrid higher-order method for convex minimization problems

This talk proposes a convergent adaptive mesh-refining algorithm for hybrid high-order methods in convex minimization problems with two-sided p-growth. Examples include the p-Laplacian, an optimal design problem in topology optimization, and the convexified double-well problem. The hybrid high-order method utilizes a gradient reconstruction in the space of piecewise Raviart-Thomas finite element functions without stabilization on triangulations into simplices. The main results imply the convergence of the energy and, under further convexity properties, of the approximations of the primal resp. dual variable. Numerical experiments illustrate an efficient approximation of singular minimizers and improved convergence rates for higher polynomial degrees. Computer simulations provide striking numerical evidence that an adopted adaptive HHO algorithm can overcome the Lavrentiev gap phenomenon even with empirical higher convergence rates.

This is joint work with Carsten Carstensen (Humboldt University of Berlin).

Tabea Tscherpel (TU Darmstadt)

Stability properties of the L^2 -projection on graded meshes

The L^2-projection mapping to Lagrange finite element spaces is an important tool in numerical analysis. Its Sobolev stability plays an important role in discrete stability and quasi-optimality estimates for parabolic problems. For adaptively generated meshes the proof of Sobolev stability is challenging and requires conditions on how strongly the mesh size varies.

We discuss stability properties under certain conditions on the polynomial degree, on the space dimension and on the mesh grading.

This is joint work with Lars Diening (Bielefeld University) and Johannes Storn (Leipzig University).